

# Regularity for a Local-Nonlocal Transmission Problem

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## Abstract

We formulate and study an elliptic transmission-like problem combining local and nonlocal elements. Let  $\mathbb{R}^n$  be separated into two components by a smooth hypersurface  $\Gamma$ . On one side of  $\Gamma$ , a function satisfies a local second-order elliptic equation. On the other, it satisfies a nonlocal one of lower order. In addition, a nonlocal transmission condition is imposed across the interface  $\Gamma$ , which has a natural variational interpretation. We deduce the existence of solutions to the corresponding Dirichlet problem, and show that under mild assumptions they are Hölder continuous, following the method of De Giorgi. The principal difficulty stems from the lack of scale invariance near  $\Gamma$ , which we circumvent by deducing a special energy estimate which is uniform in the scaling parameter. We then turn to the question of optimal regularity and qualitative properties of solutions, and show that (in the case of constant coefficients and flat  $\Gamma$ ) they satisfy a kind of transmission condition, where the ratio of “fractional conormal derivatives” on the two sides of the interface is fixed. A perturbative argument is then given to show how to obtain regularity for solutions to an equation with variable coefficients and smooth interface. Throughout, we pay special attention to a nonlinear version of the problem with a drift given by a singular integral operator of the solution, which has an interpretation in the context of quasigeostrophic dynamics.

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## 1 Introduction

The quasigeostrophic system modeling large-scale atmospheric motion induced by the rotation of the Earth is given on a half-space in  $\mathbb{R}^3$  by

$$\begin{cases} D_t L\Psi = 0 & \mathbb{R}^2 \times [0, \infty) \times [0, T) \\ D_t \partial_z \Psi = T\Psi & \mathbb{R}^2 \times \{0\} \times [0, T). \end{cases} \quad (1.1)$$

Here  $\Psi$  is the stream function, from which we can obtain the fluid velocity  $\mathbf{b} = (-\partial_y \Psi, \partial_x \Psi)$ . Then  $D_t = \partial_t + \mathbf{b}_x \partial_x + \mathbf{b}_y \partial_y$  stands for the material derivative (measuring time variation in a frame moving along the flow). The remaining quantities are  $L$ , a uniformly elliptic operator with smooth coefficients depending only on  $z$ , and the differential operator  $T$ , which we discuss below. This system is obtained by starting with the compressible Navier-Stokes equations in the Boussinesq approximation for a rotating fluid subject to constant gravity (meaning the density variation contributes only to the gravity terms). Then under the assumption that the Rossby number, which measures the ratio of the fluid's time scale to the Earth's rotation, is comparable to the ratio of the fluid's and planet's length scales, and both are small, the zero-order limiting behavior is known as the geostrophic approximation, and relates the fluid velocity to the pressure. The first-order correction gives the first equation in (1.1), while the second equation comes from considering the Ekman layer to properly account for viscosity. See [24, 15, 5] for derivations and detailed discussion.

We will assume that  $L = \Delta$  and that the initial conditions for the first equation are  $\Delta\Psi(x, y, z, 0) = 0$ ; if  $\Psi$  is sufficiently regular this reduces the equation to  $\Delta\Psi = 0$ . This reduction is more commonly studied in the mathematical literature than the full system above, especially in the case of  $T\Psi = -\Delta_{x,y}\Psi = \partial_{zz}\Psi$ . This is the form  $T$  takes when

the lower boundary is assumed to be fixed, so impervious to pressure variations. Then the system simplifies to a single equation in two dimensions (with  $s = \frac{1}{2}$  the physical case):

$$\begin{cases} \partial_t u + \langle \mathbf{b}, \nabla u \rangle + (-\Delta)^s u = 0 & x \in \mathbb{R}^2 \\ \mathbf{b} = (-R_2 u, R_1 u). \end{cases}$$

Here  $\vec{R} = (R_1, R_2)$  are the Riesz transforms and we have used the extension interpretation of  $(-\Delta)^{1/2}$ . The reason to include the parameter  $s$  is largely mathematical, as the problem is easier to solve for larger  $s$ . Indeed, the case of  $s > \frac{1}{2}$  is subcritical, and global well-posedness was known for some time; see [20, 25, 12, 10, 7] and the references therein. The critical and most interesting case of  $s = \frac{1}{2}$  proved more challenging, and the first proofs of global well-posedness were given independently in [8, 22], and since then others were found in [21, 11]. There has also been extensive work done on slightly supercritical cases [28, 13], conditional results for the critical case [12, 16], and equations with similar properties [17, 9], just to give some examples.

Now assume that  $\mathbb{R}^2$  is separated by  $\Gamma$  into components  $\Omega_2, \Omega_1$ , with  $\Omega_2$  representing land and  $\Omega_1$  ocean. The operator  $T$  comes from frictional forces near the surface  $\{z = 0\}$ , and so takes different forms in these two environments. Over a lower surface which doesn't respond to pressure variation (like the land in  $\Omega_2$ ), the generally accepted form for  $T$  can be derived from considering the Ekman layer near  $z = 0$ , and is given by  $T\Psi = -\Delta_{x,y}\Psi = \partial_{zz}\Psi$  as mentioned above. On the other hand, over a flexible boundary (like the water in  $\Omega_1$ ), the analysis appears more difficult due to the response of the frictional layer to pressure variation, and is generally proportional to  $\langle \hat{z}, \text{curl} \tau(x, y) \rangle$  where  $\tau$  represents the stress exerted on the fluid. Note that this effect comes not from the actual variations of water level (these are insignificant relative to the length scale in the model) but from the associated pressure balance and its effect on the friction forces. This stress is determined by small scale dynamics near the water, and so it can not be determined from the quantities in the model. However, one proposed approximation is that the wind forces responsible for the stress replicate the large-scale dynamics of the system (see [24, Chapter 4] for discussion), and so from the continuity of stress and velocity across the boundary we get (up to constants of proportionality) that  $\tau_x = \partial_z \mathbf{b}_x$ ,  $\tau_y = \partial_z \mathbf{b}_y$ , and  $T\Psi = \Delta_{x,y} \partial_z \Psi$ . Setting  $u = \partial_z \Psi$ , we obtain the system

$$\begin{cases} \partial_t u + \langle \mathbf{b}, \nabla u \rangle - \Delta u = 0 & x \in \Omega_1 \\ \partial_t u + \langle \mathbf{b}, \nabla u \rangle + (-\Delta)^{1/2} u = 0 & x \in \Omega_2 \\ \mathbf{b} = (-R_2 u, R_1 u). \end{cases} \quad (1.2)$$

This does not fully specify the behavior of  $u$ ; that would require conditions on how  $u$  behaves near  $\Gamma$  that are not contained in the above analysis. However, the *weak form* of the quasigeostrophic system automatically imposes this extra condition, and reduces to

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_t u \phi + \int_{\Omega_1} \langle \nabla u, \nabla \phi \rangle + \int_{\Omega_2} \int_{\Omega_2} \frac{[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^3} dy dx \\ + 2 \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)]\phi(x)}{|x - y|^3} dy dx - \int u \langle \mathbf{b}, \nabla \phi \rangle = 0 \end{aligned}$$

We actually consider the more general model

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_t u \phi + \int_{\Omega_1} \langle \nabla u, \nabla \phi \rangle + \int_{\Omega_2} \int_{\Omega_2} \frac{[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^3} dy dx \\ + \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)][v_1 \phi(x) - v_2 \phi(y)]}{|x - y|^3} dy dx - \int u \langle \mathbf{b}, \nabla \phi \rangle = \int f \phi \end{aligned} \quad (1.3)$$

for parameters  $v_1 \in (0, \infty)$  and  $v_2 \in [0, \infty)$ . The term with  $v_1$  corresponds to the nonlocal diffusive effects of  $u$  over water on  $u$  over land, and vice versa for the term with  $v_2$ . The function  $f$  represents a forcing term. The effect of  $v_1$ , as we will see, is very substantial and qualitatively changes the shapes of solutions. As will be discussed in Section 7, the term with  $v_2$  appears to be lower-order, but to exploit the variational structure of the equation most of our results will only apply to the case of  $v_1 = v_2$ . This will be assumed from now on.

The goal of the work to follow is to develop a satisfactory mathematical theory for a stationary version of this equation, and study its qualitative properties near the interface. The reason only the stationary case will be considered is that the time-dependent problem appears substantially more difficult, as will be explained. The outcome is the following theorem, which is proved in Section 9.7; admissible solutions are ones obtained as vanishing-viscosity limits, and will be defined in Definition 3.5.

**Theorem 1.1.** *There exists a unique admissible stationary solution  $u$  of (1.3), for  $\Gamma \in C^{1,1}$  globally (uniformly) and satisfying Condition 5.1. Assume  $f \in C_c^\infty$  and  $v_1 > 0$ . Then  $u$  is in  $C^{0,\gamma}(\mathbb{R}^n)$  for every  $\gamma < \alpha_0$ , where  $\alpha_0$  depends only on  $v_1$  and  $\|f\|_{L^\infty \cap L^1}$ . Moreover,  $u \in C^{1,\gamma}(\Omega_1)$ .*

Before explaining how this theorem is obtained, however, we would like to draw an analogy to transmission problems, which will guide our exposition. Classical elliptic transmission problems have been studied exhaustively from a variety of perspectives, both for their intrinsic mathematical interest (as model partial differential equations with discontinuous coefficients) and their many applications and interpretations. As a simple example, consider  $\Gamma \subset \mathbb{R}^n$  a hypersurface separating  $\mathbb{R}^n$  into two components  $\Omega_1$  and  $\Omega_2$ . For  $A_1, A_2$  two  $n \times n$  symmetric strictly positive definite matrices, we are tasked with finding minimizers to the energy

$$E[u] = \int_{\Omega_1} \langle A_1 \nabla u, \nabla u \rangle + \int_{\Omega_2} \langle A_2 \nabla u, \nabla u \rangle \quad (1.4)$$

among, say,  $\{u \in H^1(\mathbb{R}^n) | u = u_0 \text{ on } \partial\Omega\}$  for some smooth bounded domain  $\Omega$  and  $u_0 \in C^\infty(\partial\Omega)$ , where  $\langle \cdot, \cdot \rangle$  represents the Euclidean inner product. This problem can be interpreted, for instance, in terms of finding an electric potential over a region consisting of two homogeneous materials with differing dielectric constants separated by the interface  $\Gamma$ .

There are many possible approaches to studying the solutions to this problem. On one hand, the situation falls within the scope of the theory of divergence-form elliptic equations with bounded measurable coefficients, meaning energy estimates and the De Giorgi-Nash-Moser Harnack inequality are available immediately. On the other hand,

the Euler-Lagrange equation for this problem is a distributional form of the following PDE:

$$\begin{cases} \text{Tr} A_1 D^2 u = 0 & x \in \Omega_1 \\ \text{Tr} A_2 D^2 u = 0 & x \in \Omega_2 \\ \langle A_1 \nabla_{\Omega_1} u, n \rangle = \langle A_2 \nabla_{\Omega_2} u, n \rangle & x \in \Gamma \end{cases} \quad (1.5)$$

where  $\nabla_{\Omega_i} u$  means gradient evaluated from  $\Omega_i$  and  $n$  is the outward unit normal to  $\Omega_1$ . This was known early in the development of weak solution methods for elliptic equations; see [29]. The third line is known as the *transmission condition*, and demands that the conormal derivatives of  $u$  on both sides of  $\Gamma$  have a fixed ratio. Conversely, a sufficiently smooth (except in the conormal direction on  $\Gamma$ ) solution to (1.5) satisfying the transmission condition will be a minimizer to a corresponding variational problem. The precise behavior of solutions to (1.5) can be deduced relatively easily from standard theory, for instance by flattening the boundary, rewriting as an elliptic system, and applying the results in [1, 2].

Consider now a simple local-nonlocal version of the above variational problem, where the energy is given by:

$$E[u] = \int_{\Omega_1} |\nabla u|^2 + \int_{\Omega_2} \int_{\Omega_2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy + \nu \int_{\Omega_2} \int_{\Omega_1} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx, \quad (1.6)$$

Here  $\Omega_1$  and  $\Omega_2$  are smooth open sets partitioning the entire  $\mathbb{R}^n$ , and the minimization is performed over  $\{u \in H^s(\mathbb{R}^n) \cap H^1(\Omega_1) | u = u_0 \text{ on } \Omega^c\}$ . The first term is the Dirichlet energy on  $\Omega_1$ , while the second is the Gagliardo norm for  $H^s(\Omega_2)$ . The third should be interpreted as a nonlocal transmission term, placing a second, fractional-order constraint on the behavior of minimizers across the interface. When we discuss the structure of solutions near  $\Gamma$ , we will prove a more intuitive characterization of the transmission in terms of the parameter  $\nu$ , which will explain the effect of this extra term.

Existence and uniqueness for the Dirichlet problem for this (simplified) example are straightforward consequences of the uniform convexity of the energy in  $u$ , and solutions satisfy the following weak-form Euler-Lagrange equation:

$$\begin{aligned} \int_{\Omega_1} \langle \nabla u, \nabla \phi \rangle + \int_{\Omega_2} \int_{\Omega_2} \frac{[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ + \nu \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx = 0 \end{aligned} \quad (1.7)$$

for all  $\phi \in C_c^\infty(\Omega)$ . Our first task, then, is to show that solutions to this equation are continuous. For the classical version of the problem, that was an immediate application of the De Giorgi-Nash estimate [14], but this doesn't apply to our nonlocal energy.

There have recently been several efforts to bring De Giorgi or Moser iteration methods to the nonlocal framework. This is done most famously in [8], where just such an iteration is applied to prove global well-posedness of the critical SQG. Generally speaking, two ingredients are needed to apply the method of De Giorgi: some form of localized energy estimate, and invariance of the equation (or at least of the class of

equations considered) under dilation. Equation (1.7) readily admits an energy inequality (obtained by setting  $\phi = u - u_0$ , for example), but this proves insufficient. First, it needs to be localized to be useful in the iteration. In [8], a localization was performed with the help of the extension property of the fractional Laplacian; both for the sake of improved generality and simplicity of proofs, we instead follow the localization performed in [6]. Their approach uses functions of the form  $\phi = (u - \psi)_+$  in (1.7) for some suitably chosen  $\psi$  that grows fast enough as  $|x| \rightarrow \infty$  to control the tails of the rescaled  $u$ .

There is a second, and more substantial, obstruction to applying De Giorgi's argument: our equation lacks scale invariance. Indeed, as one zooms in on a point in  $\Gamma$ , the local term over  $\Omega_1$  has a higher order than the nonlocal terms, and so becomes more and more dominant. As a result, the local energy estimate always gives good control of  $\nabla u$  over  $\Omega_1$ , but a worse and worse bound on the nonlocal energy over  $\Omega_2$ . To overcome this, we prove a second, less standard, energy inequality which uses  $w = u - R[u]$  as a test function, where  $R[u]$  represents (in the case of  $\Gamma$  a hyperplane) the even reflection of  $u|_{\Omega_1}$  across  $\Gamma$ . The function  $w$  has the advantage of being supported on  $\Omega_2$ , and so the local term in (1.7) drops out, no longer obscuring the smaller nonlocal terms we need to study. In the end, we manage to estimate both  $u|_{\Omega_1}$  and the amount by which  $u$  exceeds  $R[u]$  in a scale-invariant way, which suffices to prove continuity of solutions.

The approach above can be motivated in terms of the following example in the "classical" situation, which we describe heuristically. Consider once again our first variational problem (1.4), but now with a parameter  $\epsilon \rightarrow 0$ :

$$E[w] = \int_{\Omega_1} \langle A_1 \nabla w, \nabla w \rangle + \epsilon \int_{\Omega_2} \langle A_2 \nabla w, \nabla w \rangle.$$

Let  $u_\epsilon$  be the minimizer. While the De Giorgi-Nash estimate works for each  $\epsilon$  to give that  $u_\epsilon$  is continuous, the ellipticity ratio (of order  $\epsilon^{-1}$ ) deteriorates as  $\epsilon \rightarrow 0$ , meaning the modulus obtained this way is not uniform. On the other hand, as  $\epsilon \rightarrow 0$ , we expect (if uniform estimates were available) that  $u_\epsilon$  converges to a solution of

$$\begin{cases} \text{Tr} A_1 D^2 u = 0 & x \in \Omega_1 \\ \text{Tr} A_2 D^2 u = 0 & x \in \Omega_2 \\ \langle A_1 \nabla_{\Omega_1} u, n \rangle = 0 & x \in \Gamma, \end{cases}$$

which admits a unique continuous solution obtained by solving the Neumann problem over  $\Omega_1$  and then using that solution as data for the Dirichlet problem on  $\Omega_2$ . Indeed, the regularity for this limiting problem is easily deduced by first noting that  $R[u]$  solves an equation with smooth coefficients, and then that  $u - R[u]$  is in  $H_0^1(\Omega_2)$ , and so readily admits energy estimates. Using this as motivation, one may then try doing the same for each function  $u_\epsilon$ , and this gives uniform energy estimates analogous to the ones we will derive for the nonlocal problem.

Recall that our motivating problem (1.3) contained a drift term, so to treat it we must deal with the more general equation (satisfied by solutions  $u$  for every  $\phi \in$

$C_c^\infty(\Omega) :$

$$\begin{aligned} \int_{\Omega_1} \langle A(x) \nabla u, \nabla \phi \rangle + \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[u(x) - u(y)]a(x, y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ - \int u \langle \mathbf{b}, \nabla \phi \rangle = \int f \phi. \end{aligned} \quad (1.8)$$

Here the symmetric matrix  $A(x)$  and function  $a(x, y) = a(y, x)$  are assumed to satisfy  $\lambda I \leq A(x) \leq \Lambda I$  and  $\lambda \leq a \leq \Lambda$  for positive numbers  $\lambda, \Lambda$ . This equation no longer corresponds to a minimization problem. Nevertheless, if  $s > 1/2$ ,  $\mathbf{b}$  has little effect on any of the theory developed above. If  $\mathbf{b} \in L^\infty$  and  $s = \frac{1}{2}$ , the regularity theory goes through as well. In the model, however,  $\mathbf{b} = G\vec{R}u$ , where  $\vec{R} = \nabla(-\Delta)^{-1/2}$  is the vector of Riesz transforms and  $G$  is a skew-symmetric matrix. In this case, it is easy to show that  $u \in L^\infty$ , which implies that  $\mathbf{b} \in BMO$ . This by itself does not seem to suffice for the Hölder estimate. The argument in [8], for instance, uses critically the parabolic nature of their equation, and, as discussed below, the parabolic version of our problem is substantially more difficult. Fortunately, we are still able to handle drifts which are explicitly given by Calderon-Zygmund operators of  $u$  using an argument like that in [28], re-estimating the  $BMO$  norm of  $\mathbf{b}$  in each step of the De Giorgi iteration. This leads to the following theorem (the proof of which occupies Section 4):

**Theorem 1.2.** *Let  $\Gamma$  be a Lipschitz graph on  $B_1$  with  $0 \in \Gamma$ . Assume  $u$  is an admissible solution of (1.8) on  $B_1$  with  $|u| \leq 1$  on  $\mathbb{R}^n$ . Assume further that  $\|f\|_{L^q} \leq 1$  with  $q > \frac{n}{s}$ , and either that  $\mathbf{b} = 0$ , that  $s > 1/2$ ,  $\|\mathbf{b}\|_{L^q} \leq 1$  and  $q \geq \frac{n}{2s-1}$ , or that  $s = \frac{1}{2}$ ,  $\|u\|_{L^2(\mathbb{R}^n)} \leq 1$ , and  $\mathbf{b} = G\vec{R}u$  as above. Then there is a number  $\alpha > 0$  such that*

$$[u]_{C^{0,\alpha}(B_{1/2})} \leq C_1,$$

where  $C_1, \alpha$  depend on  $\Gamma, n, s, \lambda$ , and  $\Lambda$ , and if  $s = \frac{1}{2}$ , then also on the norm of  $G$ .

After the continuity of solutions is established, the next natural topic to consider is the qualitative behavior near  $\Gamma$ . We first treat the simplest possible case of this, where we go back to equation (1.7) and assume  $\Gamma$  is a hyperplane (say  $\{x_n = 0\}$ , with  $\Omega_1 = \{x_n \leq 0\}$ ). Using a bootstrap argument and barriers, we derive the optimal regularity and asymptotic expansion for  $u$ . There is an exponent  $\alpha_0 \in ((2s-1)_+, 2s)$ , depending only on  $\nu$  and  $s$ , such that  $|u(x', x_n) - u(x', 0)| \leq C|x_n|^{\alpha_0}$ . The value of  $\alpha_0$  is explicitly computable from  $\nu$  and  $s$ . On  $\Omega_1$ , we have that, in fact,  $|u(x', x_n) - u(x', 0)| \leq C|x_n|^{\alpha_0+2-2s}$ , and moreover there is some (again explicitly computable) number  $M_0$  such that for  $(x', 0) \in \Gamma$ ,

$$\lim_{t \rightarrow 0^+} \frac{u(x', 0) - u(x', -t)}{t^{\alpha_0+2-2s}} = M_0 \lim_{t \rightarrow 0^+} \frac{u(x', 0) - u(x', t)}{t^{\alpha_0}}. \quad (1.9)$$

We will construct an explicit solution that demonstrates this is optimal, in the sense that neither side of (1.9) is 0. This property is analogous to the transmission relation in (1.5), but with different powers on the two sides of the interface. With some extra effort, a full asymptotic expansion for  $u$  can be derived in a similar way. The following theorem follows from Lemma 7.8 and Theorem 7.11:

**Theorem 1.3.** *Let  $u$  satisfy (1.7) on  $B_2$ , with  $\Gamma = \{x_n = 0\}$  as above and  $|f|, |u| \leq 1$ . Then  $u$  is smooth in the directions orthogonal to  $e_n$ , lies in  $C^{0,\alpha_0}(B_1)$  if  $\alpha_0 \leq 1$  and in  $C^{1,\alpha_0-1}(B_1)$  if  $\alpha_0 > 1$ , and also is in  $C^{1,\alpha_0+1-2s}(B_1 \cap \{x_n \leq 0\})$ . Lastly, there is a number  $l \in \mathbb{R}$  such that*

$$u(0, x_n) - u(0, 0) = l \left[ (x_n)_+^{\alpha_0} + M_0 (-x_n)_+^{\alpha_0+2-2s} \right] [1 + q(x_n)],$$

where  $|q(t)| \leq C|t|^\beta$  for some  $C, \beta$  independent of  $u$ .

The next generalization is to equations with translation invariance and flat boundary, but where the coefficients are not identically one like above. For the classical transmission problem, this is hardly more general: the jump condition is now on the conormal derivatives, as we saw above. The nonlocal version proves somewhat more subtle. The problem takes the form

$$\begin{aligned} \int_{\Omega_1} \langle A \nabla u, \nabla \phi \rangle + \int_{\Omega_2} \int_{\Omega_2} \frac{[u(x) - u(y)] a_{s,1}(x-y) [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ + \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)] a_{s,2}(x-y) [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx = 0. \end{aligned} \quad (1.10)$$

where the  $a_{s,i}$  are symmetric. Some regularity of  $a_{s,i}$  needs to be imposed to get any meaningful improvement over the De Giorgi estimate. Roughly speaking, we assume that  $a_{s,i}(z)$  are Lipschitz along rays departing from the origin. This is enough to justify a barrier construction, which allows us to prove results analogous to the above provided  $\alpha_0 < 3 - 2s$ ; here  $\alpha_0$  is computed from weighted spherical averages of the limits  $a_{s,i}^{(0)}(\hat{z}) = \lim_{t \rightarrow 0^+} a_{s,i}(t\hat{z})$ . The direction  $e_n$  is replaced by the conormal direction  $Ae_n$  on  $\Omega_1$ , and on  $\Omega_2$  by another direction  $v^*$  computed from spherical averages and moments of the limits  $a_{s,i}^{(0)}$ . In the case that  $\alpha_0 \geq 3 - 2s$ , it appears an extra structural condition on  $a_i^{(0)}$  and  $A$  needs to be satisfied for  $u$  to actually have the expected behavior at  $\Gamma$ . We call this condition *compatibility*. The theorem below is an analogue of Theorem 1.3 in this setting; see Definition 8 for the precise definition of compatibility and Definition 8.1 for the meaning of the assumptions on  $a$ . For the proof, see Theorem 8.8.

**Theorem 1.4.** *Assume  $\Gamma$  is as in Theorem 1.3. Let  $u$  solve (1.10) on  $B_2$  with  $a_{s,i} \in \mathcal{L}_2 \cap \mathcal{L}_1^*$ .*

1. *Then  $u \in C^{0,\alpha}(B_1 \cap \bar{\Omega}_2)$  for every  $\alpha < \alpha_0$ ,  $u \in C^{1,\alpha}(B_1 \cap \bar{\Omega}_1)$  for every  $\alpha < \min\{\alpha_0 + 1 - 2s, 2 - 2s\}$ , and  $\partial_{Ae_n} u(x', 0^-) = 0$  for  $|x'| < 1$ .*
2. *Moreover, if  $\alpha_0 > 1$ , then  $u \in C^{1,\alpha}(B_1 \cap \bar{\Omega}_2)$  for each  $\alpha < \min\{\alpha_0 - 1, 2 - 2s\}$  and  $\partial_{v^*} u(x', 0) = 0$ .*
3. *If in addition  $a_{s,i}, A$  are compatible,  $u \in C^{1,\alpha}(B_1 \cap \bar{\Omega}_1)$  for every  $\alpha < \alpha_0 + 1 - 2s$  and  $u \in C^{1,\alpha}(B_1 \cap \bar{\Omega}_2)$  for each  $\alpha < \alpha_0 - 1$ .*

We then discuss how to handle the case of variable coefficients and non-flat interfaces. The method involves a straightening procedure followed by a Schauder-type argument. There are two possible approaches to the Schauder theory: one based on the



method of Campanato, and another in a more localized improvement-of-flatness spirit. The Campanato method is generally simpler and is enough for most purposes, except when the compatibility of the frozen-coefficient equation is used. The other approach uses a somewhat more complicated  $L^\infty$  approximation estimate, the principal difficulty being that even in the simplest situations the solution to the constant-coefficient equation can be rather rough on  $\Omega_2$ . An application of this method proves Theorem 1.1.

We conclude with a pair of questions that we leave open. The first concerns the continuity of weak solutions to the parabolic version of this equation. Existence of solutions satisfying energy estimates is straightforward, as is the regularity when none of the parameters depend on time (this is explained below). However, in the case of time-dependent coefficients (and especially time-dependent drift, which appears in the quasigeostrophic problem above), the analysis appears substantially harder. Not only does the equation lack scale invariance, but also the natural time scales differ on  $\Omega_1$  and  $\Omega_2$ , meaning there is no clear generalization of our special energy estimates.

The other question is the case of a small  $\nu \leq 0$  in equation (1.7). In the case  $s < \frac{1}{2}$ , this presumably leads to discontinuous and possibly non-unique solutions, but if  $s > \frac{1}{2}$ , as a consequence of fractional Hardy inequality there should be a unique solution with finite energy. A heuristic analysis of the construction we use for the constant-coefficient estimate suggests that there is a continuous solution with finite energy to the weak formulation at least for  $\phi$  supported away from  $\Gamma$ . However, the De Giorgi argument we rely on so heavily used  $\nu > 0$  in an essential way, so it can not be applied here.

## 2 Notation and Definitions

We will use standard notation for Lebesgue, Hölder, and Sobolev spaces. Occasionally, when no ambiguity is possible, the notation  $C^\alpha$  will be used to refer to  $C^{[\alpha], \alpha - [\alpha]}$  where  $[\alpha]$  denotes the greatest integer below  $\alpha$ . The letter  $C$  will be reserved for constants and may change values from line to line. When important, the independence of  $C$  on some parameter will be explicitly noted. Other letters (e.g.  $C_0, C_1$ ) will be used for constants whose values are important for subsequent arguments, but may be reused in later sections.

### 2.1 Basic Definitions

Let  $\Gamma \subset \mathbb{R}^n$  be a connected locally Lipschitz submanifold of codimension 1, separating  $\mathbb{R}^n$  into two open, disjoint domains denoted by  $\Omega_1$  and  $\Omega_2$ . We will always assume that  $0 \in \Gamma$  and that locally we have that  $\Gamma$  is a Lipschitz graph, with  $\Gamma \cap \{(x', x_n) \mid |x'| < 5, |x_n| < 10L\} = \{(x', x_n) \mid |x'| < 5, x_n = g(x')\}$  for a Lipschitz function  $g$  with Lipschitz constant  $L_0$  and  $L^2 = L_0^2 + 1$  (with the convention  $\Omega_1 \cap \{(x', x_n) \mid |x'| < 5, -10L < |x_n| < 10L\} = \{(x', x_n) \mid |x'| < 5, -10L < x_n < g(x')\}$ ). There is usually no loss of generality, as this can always be obtained after a translation, dilation, and rotation depending only on the local Lipschitz character of  $\Gamma$ . We will use the notation  $E_r = \{|x_n| < 2Lr, |x'| < r\}$  for the cylinder, which will often be more convenient to work with than the ball for technical reasons. See Figure 2.1 for an illustration.

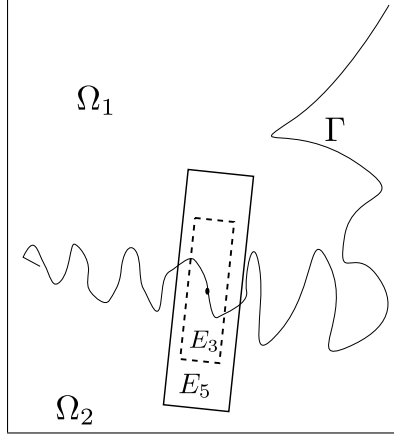


Figure 2.1: This is the situation described, with  $\Gamma$  given locally as a graph near the distinguished point 0. The cylinders  $E_r$  will only intersect  $\Gamma$  on their lateral sides provided  $r \leq 5$ .

For  $A : \Omega_1 \rightarrow \mathbb{S}^n$  measurable (where  $\mathbb{S}^n$  are the symmetric matrices in  $n$  variables) and uniformly elliptic in the sense of

$$\lambda|\xi|^2 \leq \langle A\xi, \xi \rangle \leq \Lambda|\xi|^2$$

for all  $\xi \in \mathbb{R}^n$  and some constants  $0 < \lambda < \Lambda < \infty$ , we define the bilinear form

$$B_L[u, v] = \int_{\Omega_1} \langle A \nabla u, \nabla v \rangle$$

for  $u, v \in H^1(\Omega_1)$ .

We say that a symmetric measurable function  $a : \mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1) \rightarrow [0, \Lambda]$  is *uniformly elliptic* if

$$\lambda \leq a(x, y) \quad \forall x \in \Omega_2, y \in \mathbb{R}^n,$$

and for  $s \in (0, 1)$  fixed, define the nonlocal bilinear form

$$B_N[u, v] = \int \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{[u(x) - u(y)]a(x, y)[v(x) - v(y)]dx dy}{|x - y|^{n+2s}}$$

for  $u, v \in H^s(\mathbb{R}^n)$ .

For  $f \in L^2(\mathbb{R}^n)$ ,  $\mathbf{b} \in [L^2(\mathbb{R}^n)]^n$  with  $\operatorname{div} \mathbf{b} = 0$  in the distributional sense, we say that a function  $u \in H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  is a weak solution of  $(P)$  on a domain  $\Omega$  if for every  $\phi \in C_c^\infty(\Omega)$  we have

$$B_L[u, \phi] + B_N[u, \phi] = \int_{\mathbb{R}^n} f\phi + u \langle \mathbf{b}, \nabla \phi \rangle. \quad (2.1)$$

We note that  $(P)$  is not scale-invariant: there is no scaling that preserves the equation. Nevertheless, in our efforts to prove regularity of  $(P)$ , we will need to work with

the rescaled version. We will say  $w$  solves the rescaled equation  $(P_\epsilon)$  in  $\Omega$  if for every  $\phi \in C_c^\infty(\Omega)$  we have

$$B_L^\epsilon[w, \phi] + \epsilon^{2(1-s)} B_N^\epsilon[w, \phi] = \epsilon^2 \int_{\mathbb{R}^n} f^\epsilon \phi + \epsilon \int_{\mathbb{R}^n} w \langle \mathbf{b}^\epsilon, \nabla \phi \rangle, \quad (2.2)$$

where  $f^\epsilon(x) = f(\epsilon x)$ ,  $\mathbf{b}^\epsilon(x) = \mathbf{b}(\epsilon x)$ ,

$$B_L^\epsilon[u, v] = \int_{\Omega_1/\epsilon} \langle A(\epsilon x) \nabla u(x), \nabla v(x) \rangle dx,$$

and

$$B_N^\epsilon[u, v] = \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1/\epsilon \times \Omega_1/\epsilon)} \frac{[u(x) - u(y)]a(\epsilon x, \epsilon y)[v(x) - v(y)]dx dy}{|x - y|^{n+2s}}.$$

If  $u$  solves  $(P)$  on  $\Omega$ , then  $u(\epsilon x)$  solves  $(P_\epsilon)$  on  $\Omega/\epsilon$ .

## 2.2 Tools

Next we mention some tools which will be useful in later sections; the proofs may be found in the appendix. We will require the following lemma about Sobolev extension operators on Lipschitz domains.

**Lemma 2.1.** *There is a bounded linear operator  $T : V \rightarrow H_0^1(E_1)$ , where  $V$  is the closure of  $\{u \in C^\infty(\Omega_1 \cap E_1) : u|_{\partial E_1 \cap \Omega_1} = 0\}$  in  $H^1(\Omega_1 \cap E_1)$ , with  $\|T\|$  depending only on  $L$ , satisfying the following properties:*

1.  $Tv|_{\Omega_1 \cap E_1} = v$  a.e.
2. If  $v \geq 0$ ,  $Tv \geq 0$ .
3. For every  $v \in V$  and  $l > 0$ ,

$$|\{Tv > l\}| \leq \frac{4}{3} |\{v > l\} \cap \Omega_1 \cap E_1|.$$

*Notation 2.2.* We will denote this extension operator by  $R[v]$ ; note that such a construction can be carried out on any cylinder  $E_r$ .

In the case  $s = \frac{1}{2}$  it will be of interest to consider drifts given by singular integral operators of  $u$ . In the case of  $s > \frac{1}{2}$  we will prove regularity of solutions of  $(P)$  for drifts in  $L^p$  for sufficiently large  $p$ , and this combined with Calderon-Zygmund theory will immediately apply to such a nonlinear problem. On the other hand, when  $s = \frac{1}{2}$ , the critical Lebesgue space (in the sense of being scale-invariant for the problem) for  $\mathbf{b}$  is  $L^\infty$ , while the drift will generally only lie in  $BMO$ . Treating general  $BMO$  drifts presents technical challenges, and so we will use the explicit nature of the nonlinearity, together with the following lemma from harmonic analysis, whose proof follows an argument in [30].

Let  $T$  be a Calderon-Zygmund operator with kernel  $K$  satisfying the strengthened Hormander condition

$$|K(x-y) - K(x)| \leq C \frac{|y|^\gamma}{|x|^{n+\gamma}} \quad (2.3)$$

for some  $\gamma > 0$  and all  $|x| > 2|y|$ , as well as the usual cancellation

$$\int_{r<|x|<r'} K(y)dy = 0$$

and boundedness

$$|K(y)| \leq C|y|^{-n}$$

criteria. For smooth functions  $T$  is given as the principal value integral

$$Tu(x) = P.V. \int_{\mathbb{R}^n} K(x-y)u(y)dy.$$

We claim the following:

**Lemma 2.3.** *Assume  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $|u| \leq 1 + |x|^\alpha$  for some  $\alpha < \gamma$ . Then there are constants  $\{c_B\}$  such that*

$$\sup_{B \subset B_1} \frac{1}{|B|} \int_B |Tu - c_B| \leq C,$$

where  $C$  depends only on  $T$  and  $\gamma - \alpha$ .

A simple computation reveals that this Lemma immediately gives that

$$\sup_{B \subset B_1} \frac{1}{|B|} \int_B |Tu - \oint_B Tu| \leq C$$

instead, which places  $u$  in  $BMO(B_1)$ . The John-Nirenberg inequality now applies to give

$$\left\| Tu - \oint_{B_1} Tu \right\|_{L^p} \leq C_p$$

for every  $p < \infty$ .

### 3 Existence and Crude Energy Estimates

As the drift term and asymmetry in  $a$  make the problem non-variational, the questions of existence and uniqueness of solutions require attention. Moreover, energy estimates essential to the regularity theory to follow are most conveniently justified concurrently with the construction of the solutions, as they require some extra regularity of the drift to obtain an essential cancellation. This kind of issue appears frequently in nonlinear equations coming from fluid dynamics. In this section we will show how to construct solutions to an approximate problem and prove a very weak regularity property for them. Then we will use this to justify a family of energy estimates uniform in the approximation; passing to the limit we obtain weak solutions satisfying these estimates.

We say  $u$  solves  $(P^\delta)$  on  $\Omega$  if for all  $\psi \in C_c^\infty(\Omega)$  we have that

$$B_L[u, \psi] + B_N[u, \psi] + \delta \int_{\Omega} \langle \nabla u, \nabla \psi \rangle = \int_{\Omega} f \psi + u \langle \mathbf{b}, \nabla \psi \rangle.$$

Likewise define the scaled problem  $(P_\epsilon^\delta)$ .

**Lemma 3.1.** *For each  $\epsilon, \delta > 0$ ,  $\Omega$  bounded,  $f, \mathbf{b} \in L^2(\Omega)$ ,  $\operatorname{div} \mathbf{b} = 0$ ,  $u_0 \in C_{loc}^{0,1}(\mathbb{R}^n)$ ,  $|u_0(x)| \leq 1 + |x|^{s/2}$ , there exists a unique function  $u \in H^1(\mathbb{R}^n)$  such that  $u$  solves  $(P_\epsilon^\delta)$  in  $\Omega$  and  $u = u_0$  on  $\Omega^c$ .*

*Proof.* Assume  $\epsilon = 1$ ; the general case works similarly or can be deduced from scaling. We look for a  $v \in H^1(\mathbb{R}^n)$  with  $v \equiv 0$  in  $\Omega^c$  satisfying for every  $\psi \in H_0^1(\Omega)$  (extended by 0 to make sense of  $B_N$ )

$$\begin{aligned} E[v, \psi] &:= B_L[v, \psi] + B_N[v, \psi] + \delta \int_{\Omega} \langle \nabla v, \nabla \psi \rangle - \int_{\Omega} v \langle \mathbf{b}, \nabla \psi \rangle \\ &= \int_{\Omega} f \psi + \langle u_0 \mathbf{b} - \delta \nabla u_0, \nabla \psi \rangle - B_L[u_0, \psi] - B_N[u_0, \psi]. \end{aligned} \quad (3.1)$$

Observe that  $E$  is a bounded coercive bilinear form on  $H_0^1$ :

$$E[\psi_1, \psi_2] \leq C \Lambda \|\psi_1\|_{H_0^1(\Omega)} \|\psi_2\|_{H_0^1(\Omega)}$$

from fractional Sobolev embedding of  $H^1$  into  $H^s$  to estimate the nonlocal term. On the other hand, for  $\psi \in H_0^1(\Omega)$ ,

$$E[\psi, \psi] \geq \delta \int_{\Omega} |\nabla \psi|^2 - \int_{\Omega} \psi \langle \mathbf{b}, \nabla \psi \rangle$$

since the other terms are positive. But the second term, from integration by parts and the fact that  $\operatorname{div} \mathbf{b} = 0$ , is

$$\int_{\Omega} \psi \langle \mathbf{b}, \nabla \psi \rangle = - \int_{\Omega} \langle \nabla \psi, \mathbf{b} \rangle \psi = 0,$$

implying the coercivity. On the other hand, the right-hand side in (3.1) is easily seen to be a bounded linear functional on  $H_0^1(\Omega)$ , so by Lax-Milgram theorem, there is a unique  $v$  satisfying (3.1). Finally, observe that  $v$  satisfying (3.1) is equivalent to  $v + u_0$  satisfying  $(P^\delta)$ , giving the conclusion of the lemma.  $\square$

Next, an auxiliary definition: for  $\phi : \mathbb{R}^n \rightarrow [0, \infty)$  Lipschitz and growing sufficiently slowly at infinity, define

$$M(\phi) = \max \left\{ \sup_{x, y \in \mathbb{R}^n} \frac{|\phi(x) - \phi(y)|}{|x - y|}, \sup_{\mathbb{R}^n} \frac{\phi(x)}{1 + |x|^{s/2}} \right\}.$$

Functions with finite  $M$  will be used to localize the estimates below, and  $M$  can be thought of as a measure of the flatness of  $\phi$ .

**Lemma 3.2.** *The  $u$  in Lemma 3.1 satisfies the following estimate: if  $u_0 \leq \phi$  on  $\Omega^c$ ,  $M(\phi) < \infty$ , and*

$$\|\epsilon \mathbf{b}^\epsilon\|_{L^q(\Omega)} \leq S,$$

*then*

$$\begin{aligned} \int_{\Omega_1} |\nabla(u - \phi)_+|^2 + \epsilon^{2(1-s)} \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[(u - \phi)_+(x) - (u - \phi)_+(y)]^2 dx dy}{|x - y|^{n+2s}} \\ \leq C_0 \left[ (\|\epsilon^2 f^\epsilon\|_{L^q(\Omega)}^2 + M(\phi)^2) \| \{u > \phi\} \|_{L^q(\Omega)}^{\frac{q-2}{q}} + \int_{\Omega} (u - \phi)_+^2 \right], \end{aligned} \quad (3.2)$$

where  $C_0 = C_0(n, \lambda, \Lambda, \Omega, q, S)$  is independent of  $\epsilon, \delta$ , or  $\phi$ . If we allow the constant to depend on  $\epsilon$  and  $\phi$  (but not  $\delta$ ), we may further deduce

$$\|(u - \phi)_+\|_{H^s(\mathbb{R}^n) \cap H^1(\Omega_1)} \leq C(\epsilon, \phi) (\|f\|_{L^q(\Omega)} + 1). \quad (3.3)$$

*Proof.* We drop  $\epsilon$  superscripts for clarity. Observe that as  $(u - \phi)_+ \in H^1(\mathbb{R}^n)$  vanishes outside  $\Omega$ , it is a valid test function for  $(P_\epsilon^\delta)$ . This gives

$$\begin{aligned} B_L[u, (u - \phi)_+] + \epsilon^{2(1-s)} B_N[u, (u - \phi)_+] + \delta \int_{\Omega} \langle \nabla u, \nabla(u - \phi)_+ \rangle \\ = \int_{\Omega} \epsilon^2 f(u - \phi)_+ + \epsilon u \langle \mathbf{b}, \nabla(u - \phi)_+ \rangle. \end{aligned} \quad (3.4)$$

We will show how to estimate each term, starting with the first one on the left:

$$\begin{aligned} \lambda \int_{\Omega_1} |\nabla(u - \phi)_+|^2 &\leq B_L[(u - \phi)_+, (u - \phi)_+] \\ &= B_L[u, (u - \phi)_+] - B_L[\phi, (u - \phi)_+] \\ &\leq B_L[u, (u - \phi)_+] + \frac{\lambda}{2} B_L[(u - \phi)_+, (u - \phi)_+] + \frac{\Lambda}{2\lambda} \int_{\Omega} |\nabla \phi|^2 \end{aligned}$$

where the last step used Cauchy inequality. Reabsorbing the second term on the right and estimating the integral,

$$B_L[u, (u - \phi)_+] \geq \frac{\lambda}{2} \int_{\Omega_1} |\nabla(u - \phi)_+|^2 - CM(\phi)^2 \|\{u > \phi\}\|. \quad (3.5)$$

Next, the term with  $B_N$  is treated similarly:

$$B_N[u, (u - \phi)_+] = B_N[(u - \phi)_+, (u - \phi)_+] + B_N[(u - \phi)_-, (u - \phi)_+] + B_N[\phi, (u - \phi)_+].$$

The first term, using ellipticity of  $a$ , controls the integral quantity in the estimate:

$$\frac{\lambda}{2} \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[(u - \phi)_+(x) - (u - \phi)_+(y)]^2 dx dy}{|x - y|^{n+2s}} \leq B_N[(u - \phi)_+, (u - \phi)_+].$$

The second term is positive, and so can be dropped:

$$\begin{aligned}
B_N[(u - \phi)_-, (u - \phi)_+] &= \int \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{[(u - \phi)_-(x) - (u - \phi)_-(y)]a(x, y)[(u - \phi)_+(x) - (u - \phi)_+(y)]dxdy}{|x - y|^{n+2s}} \\
&= \int \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{-(u - \phi)_-(y)[a(x, y) + a(y, x)](u - \phi)_+(x)dxdy}{|x - y|^{n+2s}} \geq 0. \quad (3.6)
\end{aligned}$$

The final part is estimated as follows:

$$\begin{aligned}
B_N[\phi, (u - \phi)_+] &= \int \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{[\phi(x) - \phi(y)]a(x, y)[(u - \phi)_+(x) - (u - \phi)_+(y)]dxdy}{|x - y|^{n+2s}} \\
&\leq \frac{\lambda}{4} B[(u - \phi)_+, (u - \phi)_+] \\
&\quad + C \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2 [1_{\{u > \phi\}}(x) + 1_{\{u > \phi\}}(y)]}{|x - y|^{n+2s}} dxdy.
\end{aligned}$$

The first of these can be reabsorbed, while the second can be further estimated as

$$\begin{aligned}
&\leq C \int_{\{u > \phi\}} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy \\
&\leq CM(\phi)^2 \int_{\{u > \phi\}} \int_{\mathbb{R}^n} \frac{\min\{|x - y|^2, |x - y|^s\}}{|x - y|^{n+2s}} dxdy \\
&\leq CM(\phi)^2 |\{u > \phi\}|,
\end{aligned}$$

where the first step used that up to enlarging the domain to  $\mathbb{R}^n \times \mathbb{R}^n$ , the two terms in the integrand are the same, the second estimated  $\phi$ , and the third evaluated the inner integral to some fixed value depending only on  $s$ . To summarize, we have the following estimate on the nonlocal term:

$$B_N[u, (u - \phi)_+] \geq \frac{\lambda}{4} \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[(u - \phi)_+(x) - (u - \phi)_+(y)]^2 dxdy}{|x - y|^{n+2s}} - CM(\phi)^2 |\{u > \phi\}|. \quad (3.7)$$

Next we bound the term in (3.2) with the coefficient  $\delta$ . As we do not want  $\delta$  dependence in the final estimate, we only need to show it is the sum of a nonnegative quantity and something controlled by the right-hand side of (3.2). We proceed as for  $B_L$ :

$$\int_{\Omega} \langle \nabla u, \nabla(u - \phi)_+ \rangle \geq \frac{1}{2} \int_{\Omega} |\nabla(u - \phi)_+|^2 - |\nabla \phi|^2 1_{\{u > \phi\}} \geq -M(\phi)^2 |\{u > \phi\}|. \quad (3.8)$$

For the drift term, we exploit the divergence-free property of  $\mathbf{b}$ :

$$\begin{aligned}
\int_{\Omega} u \langle \mathbf{b}, \nabla(u - \phi)_+ \rangle &= - \int_{\Omega} \langle \nabla u, \mathbf{b} \rangle (u - \phi)_+ \\
&= - \int_{\Omega} \langle \nabla(u - \phi)_+, \mathbf{b} \rangle (u - \phi)_+ + \int_{\Omega} \langle \nabla \phi, \mathbf{b} \rangle (u - \phi)_+,
\end{aligned}$$

the first of which vanishes. The other we estimate directly to get

$$\begin{aligned} \left| \epsilon \int_{\Omega} u \langle \mathbf{b}, \nabla(u - \phi)_+ \rangle \right| &\leq M(\phi) \| (u - \phi)_+ \|_{L^2} \| \epsilon \mathbf{b} \|_{L^q(\Omega)} \| 1_{\{u > \phi\}} \|_{L^{\frac{2q}{q-2}}} \\ &\leq M(\phi)^2 \| \{u > \phi\} \|_{L^q(\Omega)}^{\frac{q-2}{q}} \| \epsilon \mathbf{b} \|_{L^q(\Omega)}^2 + \int_{\Omega} (u - \phi)_+^2. \end{aligned} \quad (3.9)$$

Finally, the  $f$  term can be estimated as follows:

$$\begin{aligned} \left| \int_{\Omega} \epsilon^2 f(u - \phi)_+ \right| &\leq \| \epsilon^2 f \|_{L^q(\Omega)} \| (u - \phi)_+ \|_{L^2} \| \{u > \phi\} \|_{L^{\frac{2q}{q-2}}} \\ &\leq \| \epsilon^2 f \|_{L^q(\Omega)}^2 \| \{u > \phi\} \|_{L^q(\Omega)}^{\frac{q-2}{q}} + \int_{\Omega} (u - \phi)_+^2. \end{aligned} \quad (3.10)$$

Putting together (3.5), (3.7), (3.8), (3.9), and (3.10), we deduce (3.2). To see (3.3), proceed as above, but notice that the coefficient in front of the term

$$\int_{\Omega} (u - \phi)_+^2$$

can be made arbitrarily small at the expense of a larger constant in front of the remaining terms. Then the left-hand side controls the  $H^s$  seminorm of  $(u - \phi)_+$  from the fractional Sobolev embedding:

$$\begin{aligned} &\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|(u - \phi)_+(x) - (u - \phi)_+(y)|^2 dx dy}{|x - y|^{n+2s}} \\ &\leq C(\Omega) \int_{\Omega_1} |\nabla(u - \phi)_+|^2 + 2 \int_{\Omega_2 \times \mathbb{R}^n} \frac{|(u - \phi)_+(x) - (u - \phi)_+(y)|^2 dx dy}{|x - y|^{n+2s}}. \end{aligned}$$

Now apply the fractional Poincaré inequality, choose the coefficient small enough so that term can be reabsorbed, and use that  $\|\{u > \phi\}\| \leq |\Omega| \leq C$ .  $\square$

*Remark 3.3.* While the attention to the precise dependence on the various quantities on the right-hand side of (3.2) may seem tedious, it will be used in many future arguments. Basically, the quantity  $\epsilon^2 \|f\|_{L^q}$  scales to become much smaller than the others, and so can generally be assumed to be tiny. Some of the other terms will be made small via choosing a very flat function  $\phi$ , thus decreasing  $M(\phi)$ , or by having control over  $(u - \phi)_+$ , thereby ensuring that the  $L^2$  term is small. If  $\phi$  is constant, the  $\mathbf{b}$  term may be omitted.

**Theorem 3.4.** Assume  $u_0$  is uniformly Lipschitz and satisfies  $|u_0| \leq \psi$  for some  $\psi$  with finite  $M(\psi)$ . Assume also that  $f, \mathbf{b} \in L^q(\Omega)$  for some  $q \geq 2$ , and

$$\| \epsilon \mathbf{b}^\epsilon \|_{L^q(\Omega)} \leq S$$

for some  $S < \infty$ . Then for every  $\epsilon > 0$  there is a  $u$  in  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  such that  $u = u_0$  on  $\Omega^c$  and  $u$  satisfies  $(P_\epsilon)$ . This function  $u$  is the  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  weak limit



of solutions to  $(P_\epsilon^\delta)$ . Moreover, it satisfies the following property: For every  $\phi$  with  $|u_0| \leq \phi$  on  $\Omega^c$  and  $M(\phi) < \infty$ ,

$$\begin{aligned} & \int_{\Omega_1} |\nabla(u - \phi)_+|^2 + \epsilon^{2(1-s)} \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[(u - \phi)_+(x) - (u - \phi)_+(y)]^2 dx dy}{|x - y|^{n+2s}} \\ & \leq C_0 \left[ \left( \|\epsilon^2 f^\epsilon\|_{L^q(\Omega)}^2 + M(\phi)^2 \right) |\{u > \phi\}|^{\frac{q-2}{q}} + \int_{\Omega} (u - \phi)_+^2 \right]. \end{aligned} \quad (3.11)$$

Here  $C_0$  is as in Lemma 3.2.

*Proof.* Again, we suppress  $\epsilon$  superscripts. Let  $u^\delta$  be the solution to  $(P_\epsilon^\delta)$  obtained from Lemma 3.1. Applying Lemma 3.2 (taking  $\phi = u_0$  and applying to  $\pm u^\delta$ ) gives

$$\|u^\delta - u_0\|_{H^s(\mathbb{R}^n) \cap H^1(\Omega_1)}^2 \leq C(\epsilon, \Omega, f, \mathbf{b}, u_0)$$

We have that  $u^\delta$  are uniformly bounded in  $H^s(\mathbb{R}^n) \cap H^1(\Omega_1)$ , and so admit a weakly convergent subsequence  $u^{\delta_k} \rightharpoonup u$ , with  $u = u_0$  on  $\Omega^c$ .

Take a function  $w \in C_c^\infty(\Omega)$  and use it as a test function for  $(P_\epsilon^{\delta_k})$ , integrating the term with  $\delta$  by parts:

$$B_L[u^{\delta_k}, w] + \epsilon^{2(1-s)} B_N[u^{\delta_k}, w] - \delta \int_{\Omega} u^{\delta_k} \Delta w = \int_{\Omega} \epsilon^2 f w + u^{\delta_k} \langle \epsilon \mathbf{b}, \nabla w \rangle.$$

Now send  $\delta \rightarrow 0$  and use the fact that  $B_L[\cdot, w] + \epsilon^{2(1-s)} B_N[\cdot, w]$  is a continuous linear functional on  $H^s(\mathbb{R}^n) \cap H^1(\Omega_1)$  to recover

$$B_L[u, w] + \epsilon^{2(1-s)} B_N[u, w] = \int_{\Omega} \epsilon^2 f w + u \langle \epsilon \mathbf{b}, \nabla w \rangle$$

in the limit. This means  $u$  is a weak solution of  $(P_\epsilon)$ .

Now we show  $u$  inherits the family of energy estimates (3.11). For each  $u^{\delta_k}$ , from combining intermediate estimates in the proof of Lemma 3.2, we obtain

$$\begin{aligned} & B_L[(u^{\delta_k} - \phi)_+, (u^{\delta_k} - \phi)_+] + \epsilon^{2(1-s)} B_N[(u^{\delta_k} - \phi)_+, (u^{\delta_k} - \phi)_+] \\ & \leq \int_{\Omega} \left[ \epsilon^2 f (u^{\delta_k} - \phi)_+ - \epsilon (u^{\delta_k} - \phi)_+ \langle \mathbf{b}, \nabla \phi \rangle + \delta |\nabla \phi|^2 \right] \\ & \quad + B_L[\phi, (u^{\delta_k} - \phi)_+] + \epsilon^{2(1-s)} B_N[\phi, (u^{\delta_k} - \phi)_+]. \end{aligned} \quad (3.12)$$

The right-hand side is easily seen to converge as  $\delta_k \rightarrow 0$ . For the left-hand side, note that  $w \mapsto B_L[w, w] + \epsilon^{2(1-s)} B_N[w, w]$  is a continuous, convex function on  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$ , and so is weakly lower semicontinuous. Up to a subsequence, we have that  $(u^{\delta_k} - \phi)_+ \rightharpoonup (u - \phi)_+$  in  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$ , thus giving

$$\begin{aligned} & B_L[(u - \phi)_+, (u - \phi)_+] + \epsilon^{2(1-s)} B_N[(u - \phi)_+, (u - \phi)_+] \\ & \leq \int_{\Omega} \epsilon^2 f (u - \phi)_+ - \epsilon (u - \phi)_+ \langle \mathbf{b}, \nabla \phi \rangle + B_L[\phi, (u - \phi)_+] \\ & \quad + \epsilon^{2(1-s)} B_N[\phi, (u - \phi)_+]. \end{aligned}$$

Proceed as in Lemma 3.2 to obtain (3.11).  $\square$

**Definition 3.5.** A solution  $u$  obtained as an  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  weak limit of solutions to  $(P_\epsilon^k)$  is called an *admissible weak solution*. In particular, the solution obtained in Theorem 3.4 is admissible.

*Remark 3.6.* It is easily seen that the energy estimate (3.11) did not require that  $u_0 \in C^{0,1}(\mathbb{R}^n)$ , rather only that  $u_0 \in H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  and  $u$  is admissible. This comment will be used frequently; for instance, frequently an energy estimate on a subdomain is needed but regularity of the solution (and hence the data outside the subdomain) is unknown.

At this point a natural question to consider is whether weak solutions to the Dirichlet problem for  $(P)$  are unique. This is in fact easy to see in the case  $\mathbf{b} = 0$  by applying the Lax-Milgram theorem, as the arguments above give that  $B_L + B_N$  is a coercive continuous bilinear form on  $H^s(\mathbb{R}^n) \cap H^1(\Omega_1)$ . When  $\mathbf{b} \neq 0$  this argument no longer suffices, and we will need to show some differentiability of the solution on  $\Omega_2$  to prove the uniqueness of admissible solutions. As this result is not essential to the regularity theory to follow, we delay it until Section 8 for translation-invariant problems and Section 9 for the more general setting. An alternative approach is to rewrite the drift term in a way that makes it a continuous bilinear form on  $H^s$ ; we avoid this as it does not work well for the nonlinear version of the problem.

## 4 Fine Energy Estimates and Local Hölder Continuity

In this section we will show that solutions of  $(P)$  admit a Hölder modulus of continuity at points in  $\Gamma \cap \Omega$ , employing the method of De Giorgi. To this end, we prove a localized oscillation improvement estimate which can then be scaled under sufficient assumptions on  $f$  and  $\mathbf{b}$ . In particular, it will be vital that the estimate is uniform in  $\epsilon$ , in the sense that it holds with the same constants for each problem  $(P_\epsilon)$  with  $\epsilon \leq 1$ .

In this section,  $L$  will always refer to  $\sqrt{1 + L_0^2}$ , where  $L_0$  is the Lipschitz constant of the function  $g$  which locally describes  $\Gamma$  as a graph of  $x' \in \mathbb{R}^{n-1}$ . We also assume from here on that either  $\mathbf{b} = 0$  or  $s \geq \frac{1}{2}$ .

Our first objective is an improved energy estimate. The main improvement we desire is a control of a Sobolev norm of  $u$  over  $\Omega_2$  which is independent of  $\epsilon$ . We will not manage to obtain such an estimate, but fortunately it turns out that bounding the amount by which  $u$  exceeds its reflection from  $\Omega_1$  to  $\Omega_2$  will be sufficient. More precisely, let  $R[v]$  be the extension operator from Lemma 2.1, associated to the domain  $E_1$ .

**Lemma 4.1.** *Assume  $u$  is an admissible solution to  $(P_\epsilon)$  on  $E_1$ ,  $M(\phi) < \infty$ ,  $f, \mathbf{b} \in L^q(\Omega)$  for  $q \geq \frac{n}{s}$ ,  $u \leq \phi$  in  $\mathbb{R}^n \setminus E_1$ , and*

$$\|\epsilon^{2s-1} \mathbf{b}^\epsilon\|_{L^q(E_1)} \leq S < \infty.$$

*Then we have that for  $h(x) = (u - \phi)_+ - R[(u - \phi)_+]$ ,*

$$\begin{aligned} & \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[h_+(x) - h_+(y)]^2}{|x - y|^{n+2s}} dx dy + \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{-h_+(y)(u - \phi)_-(x)}{|x - y|^{n+2s}} dx dy \\ & \leq C \left[ (\|\epsilon^{2s} f^\epsilon\|_{L^q(E_1)}^2 + M(\phi)^2) \|u > \phi\|^{\frac{q-2}{q}} + \int_{E_1} (u - \phi)_+^2 \right]. \end{aligned} \quad (4.1)$$

where  $C$  depends on  $n, \lambda, \Lambda, L$ , and  $S$ , but not  $\epsilon$ .

Notice the different scaling of the  $L^q$  norm of  $\mathbf{b}$  assumed here (compared to Theorem 3.4). This reflects the difference in the orders of  $B_N$  and  $\langle \mathbf{b}, \nabla \cdot \rangle$ , and is the invariant scaling for the equation restricted to  $\Omega_2$ . The second term on the left in equation (4.1) will be used later as a nonlocal analogue of the De Giorgi isoperimetric inequality; this idea was introduced in [6].

*Proof.* We omit  $\epsilon$  superscripts. Let  $u^{\delta_k}$  solve  $(P_\epsilon^{\delta_k})$  with the same data as  $u$  and  $u^{\delta_k} \rightharpoonup u$  in  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$ . Such a sequence is guaranteed from the admissibility assumption on  $u$ . Then set  $h^{\delta_k} = (u^{\delta_k} - \phi)_+ - R[(u^{\delta_k} - \phi)_+]$  and use  $h_+^{\delta_k}$  as a test function for  $(P_\epsilon^{\delta_k})$ ; note that the reflected portion is supported on  $\Omega$ . Since this function vanishes on  $\Omega_1$ , the  $B_L$  term drops out, giving

$$\epsilon^{2(1-s)} B_N[u^{\delta_k}, h_+^{\delta_k}] + \delta_k \int_{E_1} \langle \nabla u^{\delta_k}, \nabla h_+^{\delta_k} \rangle = \int_{E_1} \epsilon^2 f h_+^{\delta_k} + \epsilon u^{\delta_k} \langle \mathbf{b}, \nabla h_+^{\delta_k} \rangle.$$

We show how to estimate each term, starting with the nonlocal one. We have

$$\begin{aligned} B_N[u^{\delta_k}, h_+^{\delta_k}] &= B_N[(u^{\delta_k} - \phi)_+, h_+^{\delta_k}] + B_N[(u^{\delta_k} - \phi)_-, h_+^{\delta_k}] + B_N[\phi, h_+^{\delta_k}] \\ &= B_N[h_+^{\delta_k}, h_+^{\delta_k}] + B_N[h_-^{\delta_k}, h_+^{\delta_k}] + B_N[R[(u^{\delta_k} - \phi)_+], h_+^{\delta_k}] \\ &\quad + B_N[(u^{\delta_k} - \phi)_-, h_+^{\delta_k}] + B_N[\phi, h_+^{\delta_k}]. \end{aligned}$$

The second of these is positive, and so will be dropped. For the term with coefficient  $\delta_k$ , use

$$\begin{aligned} \int_{E_1} \langle \nabla u^{\delta_k}, \nabla h_+^{\delta_k} \rangle &= \int_{E_1} \langle \nabla(u^{\delta_k} - \phi)_+, \nabla h_+^{\delta_k} \rangle + \int_{E_1} \langle \nabla \phi, \nabla h_+^{\delta_k} \rangle \\ &\geq \frac{1}{2} \int_{E_1} \langle \nabla h_+^{\delta_k}, \nabla h_+^{\delta_k} \rangle + \int_{E_1} \langle \nabla R[(u^{\delta_k} - \phi)_+], \nabla h_+^{\delta_k} \rangle - \int_{E_1} |\nabla \phi|^2 \\ &\geq - \int_{E_1} |\nabla R[(u^{\delta_k} - \phi)_+]|^2 - \int_{E_1} |\nabla \phi|^2. \end{aligned}$$

As  $\|R[(u^{\delta_k} - \phi)_+]\|_{H^1} \leq C$  uniformly in  $\delta$  by Lemma 3.2, this term is bounded from below. Now the second term on the right:

$$\begin{aligned} \int_{E_1} u^{\delta_k} \langle \mathbf{b}, \nabla h_+^{\delta_k} \rangle &= \int_{E_1} h_+^{\delta_k} \langle \mathbf{b}, \nabla h_+^{\delta_k} \rangle + \int_{E_1} R[(u^{\delta_k} - \phi)_+] \langle \mathbf{b}, \nabla h_+^{\delta_k} \rangle + \int_{E_1} \phi \langle \mathbf{b}, \nabla h_+^{\delta_k} \rangle \\ &= - \int_{E_1} \langle \nabla R[(u^{\delta_k} - \phi)_+], \mathbf{b} \rangle h_+^{\delta_k} - \int_{E_1} \langle \nabla \phi, \mathbf{b} \rangle h_+^{\delta_k}, \end{aligned}$$

with the second line coming from the divergence-free condition. We thus have

$$\begin{aligned} B_N[h_+^{\delta_k}, h_+^{\delta_k}] + B_N[(u^{\delta_k} - \phi)_-, h_+^{\delta_k}] &\leq -B_N[R[(u^{\delta_k} - \phi)_+], h_+^{\delta_k}] - B_N[\phi, h_+^{\delta_k}] - C\delta^k \\ &\quad - \epsilon^{2s-1} \left[ \int_{E_1} \langle \nabla R[(u^{\delta_k} - \phi)_+], \mathbf{b} \rangle h_+^{\delta_k} - \int_{E_1} \langle \nabla \phi, \mathbf{b} \rangle h_+^{\delta_k} \right] + \int_{E_1} \epsilon^{2s} f h_+^{\delta_k}. \end{aligned}$$

Now note that  $R[(u^{\delta_k} - \phi)_+]$  converges weakly in  $H^1(E_1)$ , and so strongly in  $H^s(\mathbb{R}^n)$ . It follows that the functions  $h_+^{\delta_k} \rightharpoonup h_+$  weakly in  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$ , and so strongly in  $L^2(\mathbb{R}^n)$ . The same can be said for  $(u^{\delta_k} - \phi)_-$ . The map  $v \mapsto B_N[v, v]$  is convex and bounded, so it is lower semicontinuous under weak convergence; this gives that

$$B_N[h_+, h_+] \leq \liminf_{k \rightarrow \infty} B_N[h_+^{\delta_k}, h_+^{\delta_k}].$$

Passing to a subsequence, we have that the product  $0 \leq -h_+^{\delta_k}(x)(u^{\delta_k} - \phi)_-(y) \rightarrow -h_+(x)(u - \phi)_-(y)$  for almost every  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ . Then by Fatou's Lemma, we have that

$$\begin{aligned} B_N[(u - \phi)_-, h_+] &= \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{-(u - \phi)_-(x)h_+(y) + (u - \phi)_-(y)h_+(x)a(x, y)}{|x - y|^{n+2s}} dx dy \\ &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{-(u^{\delta_k} - \phi)_-(x)h_+^{\delta_k}(y) + (u^{\delta_k} - \phi)_-(y)h_+^{\delta_k}(x)a(x, y)}{|x - y|^{n+2s}} dx dy \\ &= \liminf_{k \rightarrow \infty} B_N[(u^{\delta_k} - \phi)_-, h_+^{\delta_k}]. \end{aligned}$$

On the right-hand side, each term converges to the expected limit; for the first one use that  $R[(u^{\delta_k} - \phi)_+] \rightarrow R[(u - \phi)_+]$  strongly in  $H^s(\mathbb{R}^n)$ , while the rest are straightforward. Passing to the limit, we obtain

$$\begin{aligned} B_N[h_+, h_+] + B_N[(u - \phi)_-, h_+] &\leq -B_N[R[(u - \phi)_+], h_+] - B_N[\phi, h_+] \\ &\quad - \epsilon^{2s-1} \left[ \int_{E_1} \langle \nabla R[(u - \phi)_+], \mathbf{b} \rangle h_+ - \int_{E_1} \langle \nabla \phi, \mathbf{b} \rangle h_+ \right] + \int_{E_1} \epsilon^{2s} f h_+ \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

The left-hand side controls the quantities in 4.1. We bound each of the terms on the right.

$$|I_1| \leq \frac{1}{8} B_N[h_+, h_+] + C \|(u - \phi)_+\|_{H^1(\Omega_1)}^2,$$

so the first part is reabsorbed while the second is controlled from Theorem 3.4. For the next one,

$$|I_2| \leq \frac{1}{8} B_N[h_+, h_+] + C \int_{\Omega_2 \times \mathbb{R}^n} \frac{[\phi(x) - \phi(y)]^2 [1_{\{h>0\}}(x) + 1_{\{h>0\}}(y)] dx dy}{|x - y|^{n+2s}}.$$

This integral can be controlled by  $M(\phi)^2 \|\{u > \phi\}\|$  as in the proof of Lemma 3.2, using the fact that  $\{h > 0\} \subset \{u > \phi\}$ .

$$\begin{aligned} |I_3| &\leq \|(u - \phi)_+\|_{H^1(\Omega_1)} \|\epsilon^{2s-1} \mathbf{b}\|_{L^q(E_1)} \|h_+\|_{L^{\frac{2q}{q-2}}(E_1)} \\ &\leq C_\mu \|\epsilon^{2s-1} \mathbf{b}\|_{L^q(E_1)}^2 \|(u - \phi)_+\|_{H^1(\Omega_1)}^2 + \mu \|h_+\|_{H^s}^2, \end{aligned}$$

where the last step used the lower bound on  $q$  and the fractional Poincaré inequality. The terms  $I_4$  and  $I_5$  can be estimated in the same way as the corresponding terms in Lemma 3.2, completing the argument.  $\square$

Armed with these energy inequalities, we now prove a local  $L^\infty$  estimate.

**Lemma 4.2.** *Let  $u$  be an admissible solution of  $(P_\epsilon)$  on  $E_1$  and  $\phi$  a function with  $M(\phi) \leq 1$  and  $\phi \equiv 0$  on  $E_1$ . Assume  $u \leq 1 + \phi$  on  $\mathbb{R}^n$ ,  $\|\epsilon^{2s-1}\mathbf{b}\|_{L^q} + \|\epsilon^{2s}f\|_{L^q} \leq C_0$  for some  $q > \frac{n}{s}$ . Then there is a  $\delta > 0$  depending on  $n, \Lambda, \lambda, C_0, L, q$  but not  $\epsilon$ , such that if*

$$|\{u > 0\} \cap E_1| < \delta,$$

then

$$\sup_{E_{1/2}} u \leq \frac{1}{2}.$$

*Proof.* Let  $F : \mathbb{R}^n \rightarrow [-1, 0]$  be a smooth auxiliary function satisfying:

$$\begin{cases} F(x) \equiv -1 & x \in E_{1/2} \\ F(x) \equiv 0 & x \in E_1^c \\ |\nabla F| \leq C \end{cases}$$

Also, construct  $\psi_k = 1 + \phi + [F + \frac{1}{2}(1 - 2^{-k})]_-$  (see Figure 4.1), and observe the following properties of  $\psi_k$  and the corresponding functions  $u_k = (u - \psi_k)_+$ :

- $\psi_k$  is Lipschitz, and has  $M(\psi_k) \leq C$ .
- $0 \leq \psi_k \leq 1 + \phi$ , so that in particular  $|\{u > \psi_0\}| \leq |\{u > 0\} \cap E_1| < \delta$ .
- $\psi_k \leq \frac{1}{2}$  on  $E_{1/2}$ , so  $|\{u > \frac{1}{2}\} \cap E_{1/2}| \leq \limsup_k |\{u > \psi_k\}|$ .
- If  $u_k(x) > 0$ , then  $u_{k-1}(x) > 2^{-k}$ .
- Applying Theorem 3.4 and Lemma 4.1 on  $u$  and  $\psi_k$ , we obtain that

$$\|u_k\|_{H^1(\Omega_1)}^2 + \|(u_k - R[u_k])_+\|_{H^s(\mathbb{R}^n)}^2 \leq C \left[ |\{u_k > 0\}|^{\frac{q-2}{q}} + \int u_k^2 \right] \leq C |\{u_k > 0\}|^{\frac{q-2}{q}};$$

notice that we absorbed all of the  $f, \mathbf{b}$  dependence into  $C$  and dropped some terms on the left. The last step used the fact that  $u_k \leq 1$ .

Now set  $A_k = |\{u_k > 0\}|$ . The lemma will follow if we can show that there is a  $\delta > 0$  such that if  $A_0 < \delta$ , then  $A_k \rightarrow 0$ . We will show a nonlinear recurrence relation of the form  $A_k \leq C^k A_{k-1}^\beta$  for some  $\beta > 1$ ; this implies the conclusion. Let  $v_k = \max\{u_k, R[u_k]\} = R[u_k] + (u_k - R[u_k])_+$ . Then

$$A_k \leq |\{u_{k-1} > 2^{-k}\}| \leq 2^{kp} \int u_{k-1}^p \leq C^k \int v_{k-1}^p \leq C^k \|v_{k-1}\|_{H^s(\mathbb{R}^n)}^p$$

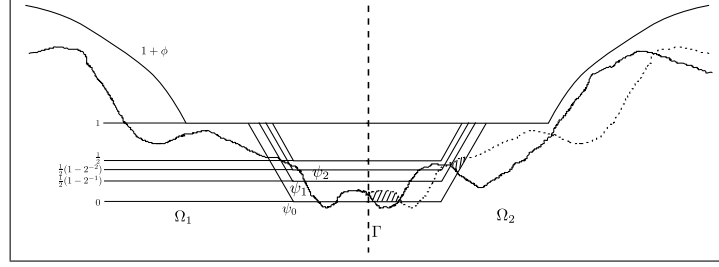


Figure 4.1: Here  $n = 1$ ,  $\Gamma$  is the point in the center of the drawing, and the vertical axis represents the values of the function  $u$ , rendered as the solid curve. Under the hypotheses of Lemma 4.2,  $u$  is guaranteed to lie beneath the graph of  $1 + \phi$ , which is shown. The first three auxilliary functions  $\psi_k$  are displayed, as well as the limiting function  $\lim_k \psi_k$ ; notice that they are truncated vertical translates of each other. Finally, the dashed curve represents the reflection  $R[u]$ , and the area of the shaded region is the integral of  $(u_0 - R[u_0])_+$ .

where  $p = \frac{2n}{n-2s}$ . The first step comes from Chebyshev's inequality, the second uses the fact that  $u_k \leq v_k$ , and the last one follows from the fractional Poincaré inequality. Using the second expression for  $v_{k-1}$ , we see that

$$A_k \leq C^k \left( \|R[u_{k-1}]\|_{H^s(\mathbb{R}^n)}^2 + \|(u_{k-1} - R[u_{k-1}])_+\|_{H^s(\mathbb{R}^n)}^2 \right)^{p/2} \leq C^k A_{k-1}^{\frac{p(q-2)}{2q}}.$$

A computation gives that  $\frac{p(q-2)}{2q} > 1$  provided  $q > \frac{n}{s}$ , and so we have the recursion as claimed.  $\square$

The next two lemmas show that the measure of the set where the oscillation of  $u$  improves is almost full. This kind of measure estimate is the second component of De Giorgi's argument [14], and will be combined with Lemma 4.2 to prove that the oscillation of  $u$  decays near  $\Gamma$ .

**Lemma 4.3.** *Let  $u$  be an admissible solution to  $(P_\epsilon)$  on  $E_3$  with  $\|\epsilon^{2s-1}b\|_{L^q(E_3)}^2 \leq C_0$  (for some  $q > \frac{n}{s}$ ) satisfying  $u \leq 1 + \phi$  on  $\mathbb{R}^n$  (where  $\phi \equiv 0$  on  $E_3$ ). Then for every  $\mu, \delta > 0$  there are  $\theta_0, \eta_0 > 0$ , depending on  $n, \lambda, \Lambda, q, L, C_0$ , but not  $\epsilon$ , such that if  $\|\epsilon^{2s}f\|_{L^q(E_3)} + M(\phi) < \eta_0$  and*

$$|\{u \leq 0\} \cap E_1 \cap \Omega_1| > \mu, \quad (4.2)$$

then

$$|\{u > 1 - \theta_0\} \cap E_2 \cap \Omega_1| < \delta. \quad (4.3)$$

*Proof.* Construct an auxiliary function  $F : \mathbb{R}^n \rightarrow [-1, 0]$  satisfying the following properties:

$$\begin{cases} F(x) \equiv -1 & x \in E_2 \\ F(x) \equiv 0 & x \in E_3^c \\ |\nabla F| \leq C \end{cases}$$

Now consider the functions  $\phi_k = 2^k(1 + \phi + 2^{-k}F)$ , which have  $M(\phi_k) \leq C\eta_0 2^k$ ,  $u_k = 2^k u$ , which are admissible solutions to  $(P_\epsilon)$  with right-hand side  $2^k f$ , and  $v_k = (u_k - \phi_k)_+$ .

**Claim.** *There is a constant  $\gamma = \gamma(n, \lambda, \Lambda, L, q, C_0, \delta, \mu) > 0$  such that provided  $\eta_0 < 2^{-k}$ , either*

$$|\{v_k > \frac{1}{2}\} \cap E_2 \cap \Omega_1| < \delta$$

or

$$|\{0 < v_k < \frac{1}{2}\} \cap E_2 \cap \Omega_1| \geq \gamma.$$

*Proof.* By the estimate in Theorem 3.4, we have that

$$\|v_k\|_{H^1(\Omega_1)}^2 \leq C \left[ (\|2^k \epsilon^2 f\|_{L^q(E_3)}^2 + M(\phi_k)^2) |\{u_k > \phi_k\}|^{\frac{q-2}{q}} + \int_{\Omega} v_k^2 \right] \leq C,$$

where we use the assumption that  $\eta_0 < 2^{-k}$ , together with the fact that  $v_k \leq 1$  and is supported on  $E_3$ , in order to estimate the right-hand side by a constant  $C$  which is independent of  $k$ . From (4.2), we have that

$$|\{v_k \leq 0\} \cap E_1 \cap \Omega_1| > \mu.$$

Assume that

$$|\{v_k > \frac{1}{2}\} \cap E_2 \cap \Omega_1| \geq \delta.$$

Then by the De Giorgi isoperimetric inequality applied to  $R[v_k]$  on the full cylinder  $E_2$  (the proof may be found in the appendix of [8]) and the level set estimate in Lemma 2.1, we have

$$\begin{aligned} \delta^{1-\frac{1}{n}} \mu &\leq |\{v_k > \frac{1}{2}\} \cap E_2 \cap \Omega_1|^{1-\frac{1}{n}} |\{v_k \leq 0\} \cap E_2 \cap \Omega_1| \\ &\leq C \|v_k\|_{H^1(E_2 \cap \Omega_1)} |\{0 < v_k < \frac{1}{2}\} \cap E_2 \cap \Omega_1|^{\frac{1}{2}}. \end{aligned}$$

This immediately gives  $|\{0 < v_k < \frac{1}{2}\} \cap \Omega_1 \cap E_2| \geq c(\delta, \mu) := \gamma$ .  $\square$

Now observe that the sets  $\{0 < v_k < \frac{1}{2}\} \cap E_2 \cap \Omega_1$  are disjoint for distinct  $k$ . Choose  $k_0 = \lceil \frac{|E_2|}{\gamma} \rceil + 1$  and  $\eta_0 < 2^{-k_0}$ . Then the claim applies for each  $0 \leq k \leq k_0$ . However, if  $|\{0 < v_k < \frac{1}{2}\} \cap E_2 \cap \Omega_1| \geq \gamma$  for each of these  $k$ , that would mean

$$|E_2| \geq \sum_{k=0}^{k_0} |\{0 < v_k < \frac{1}{2}\} \cap E_2 \cap \Omega_1| \geq |E_2| + \gamma,$$

which is absurd. It follows that for some  $k \leq k_0$ , we must have  $|\{v_k > \frac{1}{2}\} \cap E_2 \cap \Omega_1| < \delta$ . After rescaling this gives  $|\{u > 1 - 2^{-k-1}\} \cap E_2 \cap \Omega_1| < \delta$ , and so (4.3) holds with  $\theta_0 = 2^{-k_0-1}$ .  $\square$

**Lemma 4.4.** *Under the hypotheses of Lemma 4.3, for some (smaller)  $\theta_1, \eta_1 > 0$ , we have the additional conclusion that*

$$|\{u > 1 - \theta_1\} \cap E_1 \cap \Omega_2| < \delta.$$

*Proof.* This time let  $F : \mathbb{R}^n \rightarrow [-1, 0]$

$$\begin{cases} F(x) \equiv -1 & x \in E_1 \\ F(x) \equiv 0 & x \in E_2^c \\ |\nabla F| \leq C \end{cases}$$

and set  $\phi_\theta = 1 + \phi + \theta F$ . Observe that  $M(\phi_\theta) \leq C(\eta_1 + \theta) \leq C\theta$  provided  $\eta_1 < \theta$ , and so from Lemma 4.1 we have that

$$\begin{aligned} & \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{-(u - \phi_\theta)_-(x)((u - \phi_\theta)_+ - R[(u - \phi_\theta)_+])_+(y) dx dy}{|x - y|^{n+2s}} \\ & \leq C \left[ (\|\epsilon^{2s} f\|_{L^q(E_1)}^2 + M(\phi_\theta)^2) |\{u > \phi_\theta\}|^{\frac{q-2}{q}} + \int_{E_1} (u - \phi_\theta)_+^2 \right] \\ & \leq C (\eta_1^2 + M(\phi_\theta) + \theta^2) \leq C\theta^2; \end{aligned}$$

for the last term use the fact that  $(u - \phi_\theta)_+ \leq \theta$ . From (4.2) we know that  $|\{u \leq 0\} \cap E_1 \cap \Omega_1| > \mu$ , which implies that (provided  $\theta < \frac{1}{2}$ , say)

$$|\{-(u - \phi_\theta)_- \geq \frac{1}{2}\} \cap E_1| \geq \mu.$$

As  $((u - \phi_\theta)_+ - R[(u - \phi_\theta)_+])_+$  is supported on  $E_2$ , we can use a lower bound on the kernel  $|x - y|^{-n-2s} \geq c$  for  $x \in E_1, y \in E_2$  to deduce that

$$\begin{aligned} C\theta^2 & \geq \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{-(u - \phi_\theta)_-(x)((u - \phi_\theta)_+ - R[(u - \phi_\theta)_+])_+(y) dx dy}{|x - y|^{n+2s}} \\ & \geq \int_{\Omega_2 \cap E_2} \int_{\{-(u - \phi_\theta)_- \geq \frac{1}{2}\} \cap E_1} \frac{-(u - \phi_\theta)_-(x)((u - \phi_\theta)_+ - R[(u - \phi_\theta)_+])_+(y) dx dy}{|x - y|^{n+2s}} \\ & \geq c\mu \int_{E_2 \cap \Omega_2} ((u - \phi_\theta)_+ - R[(u - \phi_\theta)_+])_+ \\ & \geq c\mu\theta \left| \{(u - \phi_\theta)_+ > R[(u - \phi_\theta)_+] + \frac{\theta}{4}\} \right|, \end{aligned}$$

with the last step by Chebyshev's inequality. By choosing  $\theta$  small enough, we can guarantee that

$$\left| \{(u - \phi_\theta)_+ > R[(u - \phi_\theta)_+] + \frac{\theta}{4}\} \right| < \frac{\delta}{4}.$$

On the other hand, provided  $\theta$  is smaller than  $\frac{1}{2}\theta_0(\frac{\delta}{4}, \mu)$  (and  $\eta_1$  is chosen smaller), the conclusion of Lemma 4.3 gives that  $|\{u > 1 - \theta\} \cap E_2 \cap \Omega_1| < \frac{\delta}{4}$ . It then follows that



$|\{R[(u - \phi_\theta)_+] > 0\}| < \frac{\delta}{3}$ . Combining these,

$$\begin{aligned} |\{u > 1 - \frac{\theta}{2}\} \cap E_1 \cap \Omega_2| &\leq |\{(u - \phi_\theta)_+ > \frac{\theta}{2}\} \cap \Omega_2| \\ &\leq |\{(u - \phi_\theta)_+ > R[(u - \phi_\theta)_+] + \frac{\theta}{4}\}| + \left| \{R[(u - \phi_\theta)_+] > \frac{\theta}{4}\} \right| \\ &< \delta. \end{aligned}$$

Choose  $\theta_1 = \theta/2$  to deduce the conclusion.  $\square$

**Lemma 4.5.** *Under the hypotheses of 4.3, for some (smaller)  $\theta_2, \eta_2 > 0$ , we have the additional conclusion that*

$$\sup_{E_{1/2}} u \leq 1 - \theta_2.$$

*Proof.* Fix  $\delta$  as in Lemma 4.2 and take  $2\theta_2, \eta_2$  small enough so that both Lemma 4.3 and Lemma 4.4 apply with this  $\delta$ . Then we have that  $|\{u > 1 - 2\theta_2\} \cap E_1| < \delta$ . Let  $v = (2\theta_2)^{-1}(u - 1 + 2\theta_2)$ ; then provided  $\eta_2$  is chosen even smaller, Lemma 4.2 applies to  $v$  to give

$$\sup_{E_{1/2}} v \leq \frac{1}{2},$$

which scales to

$$\sup_{E_{1/2}} u \leq 1 - \theta_2. \quad \square$$

We are now in a position to iterate Lemma 4.5 to obtain geometric oscillation decay.

**Theorem 4.6.** *Assume  $u$  is an admissible solution of (P) on  $E_1$  with  $|u| \leq 1$  on  $\mathbb{R}^n$ . Assume further that  $\|f\|_{L^q} \leq 1$  with  $q > \frac{n}{s}$ , and either that  $\mathbf{b} = 0$  or that  $s > 1/2$ ,  $\|\mathbf{b}\|_{L^q} \leq C_0$  and  $q \geq \frac{n}{2s-1}$ . Then there are  $r, \theta > 0$  such that*

$$\text{osc}_{E_{\rho^k}} u \leq 2(1 - \theta)^k.$$

*Proof.* The conclusion is immediate from the assumptions when  $k = 0$ . Assume by induction that it holds for every  $k < l$ ; we will show it holds for  $l$ . Let

$$a_{l-1} = \frac{\max_{E_{\rho^{l-1}}} u + \min_{E_{\rho^{l-1}}} u}{2};$$

then if we set  $v(x) = \frac{u(\rho^{l-1}x) - a_{l-1}}{(1-\theta)^{l-1}}$ , we have from induction that  $|v| \leq 1$  on  $E_{\rho^{-1}}$ . Moreover,

$$\sup_{E_{\rho^j}} |v| \leq 2(1 - \theta)^{-j} - 1$$

for  $0 \leq j \leq l-1$ . By choosing  $\rho^2 = r$ , It follows that for  $|x| > \rho^{-1}$ ,

$$|v(x)| \leq 2(1 - \theta)^{-\frac{1}{2}} |x|^{\frac{\log(1-\theta)}{\log r}} - 1 := 1 + \phi. \quad (4.4)$$

Notice that  $\phi$  is radial,  $\partial_r \phi > 0$ , and  $\partial_{rr} \phi < 0$ . Thus  $|\nabla \phi(|x|)| \leq |\nabla \phi(\rho^{-1})| \leq C\alpha\rho^{1-\alpha}$ , which in particular can be made arbitrarily small by choosing  $\rho$  (and hence  $r$ ) smaller. Then provided  $\alpha = \frac{\log(1-\theta)}{\log r}$  is small enough (which, again, is arranged by choosing  $r$  smaller), we can make sure  $M(\phi)$  is arbitrarily small as well.

Next, observe that  $v$  solves  $(P_\epsilon)$  with  $\epsilon = \rho r^{l-1}$ ,  $\mathbf{b}^\epsilon(x) = \mathbf{b}(\rho r^{l-1}x)$ , and  $f^\epsilon(x) = (1-\theta)^{l-1}f(\rho r^{l-1}x)$ . Notice that  $\|\epsilon^{2s-1}\mathbf{b}^\epsilon\|_{L^q(E_3)}^2 \leq \epsilon^{2(2s-1-n/q)}C_0$ . Then if  $q \geq \frac{n}{2s-1}$ , this is bounded by  $C_0$ . Also, after an initial dilation, we may assume that

$$\|\epsilon^{2s}f^\epsilon\|_{L^q(E_3)}^2 \leq \epsilon^{2(2s-n/q)}(1-\theta)^{l-1}\|f\|_{L^q(E_1)}^2 \leq \eta$$

if  $\theta$  is small enough. Applying Lemma 4.5 with  $\mu = |E_1 \cap \Omega_1|/\epsilon/2$  to either  $v$  or  $-v$ , we obtain that if  $\eta, r$  are small enough, there is a constant  $\theta_2$  such that either

$$\sup_{E_{1/2}} v \leq 1 - \theta_2$$

or

$$\inf_{E_{1/2}} v \geq -1 + \theta_2.$$

As  $\Omega_1$  has Lipschitz constant  $L$  near 0, we have that  $|\Omega_1/\epsilon \cap E_1| \geq c(L)|E_1|$ , meaning  $\mu$  can be chosen uniformly in  $\epsilon$ . In other words,

$$\text{osc}_{E_{1/2}} v \leq 2(1 - \theta_2/2).$$

Scaling back, making sure that  $\rho < \frac{1}{6}$ , and setting  $\theta = \theta_2/2$ , we get

$$\text{osc}_{E_\rho} v \leq 2(1 - \theta)^l,$$

as desired.  $\square$

We now turn our attention to the case  $s = \frac{1}{2}$ . The above argument goes through unchanged if  $\mathbf{b} \in L^\infty$ . We are also interested, however, in the nonlinear problem where  $u$  and  $\mathbf{b}$  are related by

$$\mathbf{b} = Tu,$$

where  $T$  is a vector-valued translation invariant Calderon-Zygmund operator satisfying (2.3) and having the property that  $\text{div} Tu = 0$  (in the distributional sense). The standard example arises from the surface quasigeostrophic equation, where  $n = 2$  and  $T = (-R_2, R_1)$  with  $R_i$  Riesz transforms.

**Theorem 4.7.** *Assume  $u$  is an admissible solution of  $(P)$  on  $E_1$  with  $|u| \leq 1$  on  $\mathbb{R}^n$ . Assume further that  $s = \frac{1}{2}$ ,  $\|f\|_{L^q} \leq 1$  with  $q > \frac{n}{s}$ ,  $\mathbf{b} = Tu$  with  $T$  as above, and  $|\int_{B_1} \mathbf{b}| \leq C_0$ . Then there are  $r, \theta > 0$  such that*

$$\text{osc}_{E_{\rho^k}} u \leq 2(1 - \theta)^k$$

and

$$\sup_{B \subset B_{\rho^k}} \left| \int_B \mathbf{b} - \int_B \mathbf{b} \right| \leq C(1 - \theta)^{k-1} \quad (4.5)$$

*Proof.* We proceed as in the proof of Theorem 4.6. The base case  $k = 0$  follows immediately from applying Lemma 2.3 and using  $|u| \leq 1$ . Assume as before that the conclusion holds for  $k < l$  and set

$$v(x) = \frac{u(\rho r^{l-1}x) - a_{l-1}}{(1-\theta)^{l-1}}.$$

Note that  $v$  satisfies  $|v| \leq 1 + \phi$  with  $\phi$  as in (4.4). Moreover,  $v$  satisfies  $(P_\epsilon)$  with  $\epsilon = \rho r^{l-1}$  and

$$\mathbf{b}^\epsilon(x) = \mathbf{b}(\rho r^{l-1}x) = (1-\theta)^{l-1}Tv(x).$$

Making sure that  $r, \theta$  are small enough, we have that  $\phi \leq (|x| - \frac{1}{4\rho})_+^\alpha$  for some  $\alpha < \gamma$  (with  $\gamma$  as in (2.3)). Applying Lemma 2.3, we have that

$$\sup_{B \subset B_{(4\rho)^{-1}}} \left| \int_B \mathbf{b}^\epsilon - \int_B \mathbf{b} \right| \leq C(1-\theta)^{l-1}. \quad (4.6)$$

Next, we may apply the John-Nirenberg inequality to the inductive assumption (4.5), we see that

$$\left( \int_{B_{r^{k+1}}} \left| \mathbf{b} - \int_{B_{r^k}} \mathbf{b} \right|^p \right)^{1/p} \leq r^{-n/p} C(1-\theta)^{k-1}$$

and

$$\left( \int_{E_{3\rho r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-1}}} \mathbf{b} \right|^p \right)^{1/p} \leq C\rho^{-n/p}(1-\theta)^{l-1}$$

for  $k < l$  and  $p$  large and to be chosen below. Since the averages are being taken over subdomains, we incur the factors of  $r^{-n/p}$  on the right. In the second inequality we used the fact that  $E_3 \subset B_{\frac{1}{4\rho}}$ , which can be guaranteed by choosing  $\rho$  to be small in terms of  $L$ . Now we can compute

$$\begin{aligned} \left( \int_{E_3} |\mathbf{b}^\epsilon|^p \right)^{1/p} &\leq \left| \int_{B_{r^{l-1}}} \mathbf{b} \right| + \left( \int_{E_{3\rho r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-1}}} \mathbf{b} \right|^p \right)^{1/p} \\ &\leq \left( \int_{B_{r^{l-1}}} |\mathbf{b}|^p \right)^{1/p} + \left( \int_{E_{3\rho r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-1}}} \mathbf{b} \right|^p \right)^{1/p} \\ &\leq \left( \int_{B_{r^{l-2}}} |\mathbf{b}|^p \right)^{1/p} + \left( \int_{B_{r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-2}}} \mathbf{b} \right|^p \right)^{1/p} + \left( \int_{E_{3\rho r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-1}}} \mathbf{b} \right|^p \right)^{1/p} \\ &\leq C \left[ \left( \int_{E_{3\rho r^{l-1}}} \left| \mathbf{b} - \int_{B_{r^{l-1}}} \mathbf{b} \right|^p \right)^{1/p} + \sum_{k=1}^{l-1} \left( \int_{B_{r^k}} \left| \mathbf{b} - \int_{B_{r^{k-1}}} \mathbf{b} \right|^p \right)^{1/p} \right] + C_0 \\ &\leq C \sum_{k=1}^l r^{-n/p} (1-\theta)^{k-1} + C_0 \leq C, \end{aligned}$$

with  $C$  independent of  $l$ . Now we make sure  $p$  is large enough so Lemma 4.5 applies to give

$$\text{osc}_{E_{1/2}} v \leq 2(1-\theta),$$

and proceed as in the proof of Theorem 4.6. To obtain (4.5), rescale equation (4.6).  $\square$

*Remark 4.8.* The assumption on the  $L^q$  space that  $f$  belongs to is not optimal; the best possible range is  $q > \frac{n}{2s}$ . This improvement requires only minor changes to the proofs above, estimating the terms with  $f$  as in [18]. We chose to suppress the greater generality only to simplify the exposition. On the other hand, the Lebesgue spaces for  $\mathbf{b}$  can not be improved.

## 5 Global Results for the Elliptic Problem

In this section, we consider the general problem on  $\mathbb{R}^n$  with Lipschitz interface. We demonstrate that solutions exist and are globally bounded, and then give a procedure to flatten the interface in a neighborhood of the origin to obtain a local solution to a problem. This is made more complicated by the nonlocal nature of the equation, but nevertheless is generally possible. This flattening will be used in later sections to study the finer behavior of a solution  $u$  near a smooth stretch of interface.

We begin with existence and global bounds for the problem. The procedure is identical to that in Section 3, except now care is taken with regard to compactness issues arising from the unbounded domain. We will require the following strengthened assumption on  $\Gamma$ :

**Condition 5.1.** Both  $\Omega_1, \Omega_2$  admit a Gagliardo-Nirenberg-Sobolev inequality of the type

$$\|u\|_{L^{p_1}(\Omega_i)} \leq C_1 \|\nabla u\|_{L^2(\Omega_i)}$$

with  $p_1 = \frac{2n}{n-2}$  for  $u$  with bounded support, and

$$\|u\|_{L^{p_2}(\Omega_2)} \leq C_2 \left( \int_{\Omega_2 \times \Omega_2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}$$

for  $p_2 = \frac{2n}{n-2s}$  and  $u$  with bounded support.

These are satisfied by images of a half-space under globally bilipschitz maps, for instance. In fact, only somewhat weaker conditions are needed, but we do not pursue this point. In the lemma below, the solutions we will construct will not lie in  $L^2(\mathbb{R}^n)$  because they decay too slowly at infinity (this is an inevitable feature of the global problem). Nevertheless, they have finite energy and they embed in appropriate Lebesgue spaces via the Gagliardo-Nirenberg-Sobolev inequality above. We carry over the concept of admissibility to this situation.

**Lemma 5.2.** *Let  $f \in L^1 \cap L^2(\mathbb{R}^n)$ ,  $\mathbf{b} \in L^2(\mathbb{R}^n)$ . Assume Condition 5.1. Then there exists an admissible finite-energy solution to (P) on  $\mathbb{R}^n$ . This solution satisfies the following additional estimate, for each  $l \geq 0$ :*

$$B_L[(u - l)_+, (u - l)_+] + B_N[(u - l)_+, (u - l)_+] \leq \int_B f(u - l)_+$$

*Proof.* We construct global solutions to the approximate problem  $(P^\delta)$  with the same data. To do so, let  $u_R^\delta$  solve  $(P^\delta)$  on  $B_R$  with boundary data identically zero and  $f, \mathbf{b}$  as

right-hand side and drift; these exist by Lemma 3.1. Using  $u_R^\delta$  as a test function and proceeding as in, say, the proof of Lemma 3.2, we obtain

$$B_L[u_R^\delta, u_R^\delta] + B_N[u_R^\delta, u_R^\delta] + \delta \int_{B_R} |\nabla u_R^\delta|^2 = \int_{B_R} f u_R^\delta.$$

Estimating both sides using ellipticity and the condition on  $f$ ,

$$\begin{aligned} \int_{\Omega_1} |\nabla u_R^\delta|^2 + \int_{\mathbb{R}^n \times \Omega_2} \frac{|u_R^\delta(x) - u_R^\delta(y)|^2}{|x - y|^{n+2s}} dx dy + \delta \int |\nabla u_R^\delta|^2 \\ \leq \|f\|_{L^{\frac{p_1}{p_1-1}}(\mathbb{R}^n)} \|u_R^\delta\|_{L^{p_1}(\mathbb{R}^n)} \\ \leq C(n, \lambda, \Lambda, \nu) \|f\|_{L^1 \cap L^2(\mathbb{R}^n)}^2 + \nu \|u_R^\delta\|_{L^{p_1}(\mathbb{R}^n)}^2 \end{aligned}$$

where  $p_1 = \frac{2n}{n-2}$ . Choosing  $\nu$  to be small in terms of  $\delta$  and applying Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\|\nabla u_R^\delta\|_{L^2} + \|u_R^\delta\|_{L^{p_1}} \leq C(n, \lambda, \Lambda, \delta) \|f\|_{L^1 \cap L^2}. \quad (5.1)$$

Passing to a weak limit, we obtain a solution  $u^\delta$  to  $(P^\delta)$  on  $\mathbb{R}^n$  with the inequality

$$B_L[u^\delta, u^\delta] + B_N[u^\delta, u^\delta] + \delta \int |\nabla u^\delta|^2 \leq \int f u^\delta.$$

This implies that, if  $p_2 = \frac{2n}{n-2s}$ , then

$$\begin{aligned} \int_{\Omega_1} |\nabla u^\delta|^2 + \int_{\Omega_2 \times \Omega_2} \frac{|u^\delta(x) - u^\delta(y)|^2}{|x - y|^{n+2s}} dx dy \\ \leq C(n, \lambda, \Lambda, \nu) \|f\|_{L^1 \cap L^2}^2 + \nu \left[ \|u^\delta\|_{L^{p_1}(\Omega_1)}^2 + \|u^\delta\|_{L^{p_2}(\Omega_2)}^2 \right]. \end{aligned}$$

Choosing  $\nu$  to be small in terms of the quantities in Condition 5.1 and applying Gagliardo-Nirenberg-Sobolev inequality to the domains  $\Omega_1, \Omega_2$  we see that

$$\begin{aligned} \|u^\delta\|_{L^{p_1}(\Omega_1)} + \|\nabla u^\delta\|_{L^2(\Omega_1)} + \|u^\delta\|_{L^{p_2}(\Omega_2)} + \left( \int_{\Omega_2 \times \Omega_2} \frac{|u^\delta(x) - u^\delta(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2} \\ \leq C(n, \lambda, \Lambda, \Gamma) \|f\|_{L^1 \cap L^2}. \end{aligned}$$

This estimate is uniform in  $\delta$ , and passing to the weak limit we obtain an admissible solution  $u$  to  $(P)$ . The energy inequality also passes to the limit, as, clearly, does the following level-set version of it (for  $l \geq 0$ ):

$$B_L[(u - l)_+, (u - l)_+] + B_N[(u - l)_+, (u - l)_+] \leq \int f(u - l)_+.$$

□

This solution has a global  $L^\infty$  estimate, whose proof we give below.

**Lemma 5.3.** Assume  $f \in L^q(\mathbb{R}^n)$  with  $q > \frac{n}{2s}$ . The solution  $u$  from Lemma 5.2 is bounded, with

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n, \lambda, \Lambda, \Gamma) \|f\|_{L^q \cap L^1}.$$

Assume instead  $u$  is an admissible solution to (P) on a bounded Lipschitz domain  $\Omega$ , with bounded data  $u_0$ . Then  $u$  is bounded, with

$$\|u\|_{L^\infty(\Omega)} \leq C(n, \lambda, \Lambda, \Gamma, \Omega) [\|f\|_{L^q} + \|u_0\|_{L^\infty}]$$

*Proof.* To prove the first conclusion, we will show that there is a  $\delta > 0$  such that if  $\|f\|_{L^1 \cap L^q} \leq \delta$ , then

$$\sup_{\mathbb{R}^n} u \leq 1.$$

Then the lemma follows from scaling. Let  $l_k = 1 - 2^{-k}$ , and set

$$A_k = \|(u - l_k)_+\|_{L^{p_1}(\Omega_1)} + \|(u - l_k)_+\|_{L^{p_2}(\Omega_2)}.$$

We know that  $A_0 < \delta$  by Lemma 5.2, and will show that  $A_k \rightarrow 0$  as  $k \rightarrow \infty$ . For  $k \geq 1$ ,

$$\begin{aligned} A_k^2 &\leq C(\Gamma, n) [\|(u - l_k)_+\|_{H^s(\Omega_2)}^2 + \|(u - l_k)_+\|_{H^1(\Omega_1)}^2] \\ &\leq C(n, \lambda, \Lambda, \Gamma) \left( B_L[(u - l_k)_+, (u - l_k)_+] + B_N[(u - l_k)_+, (u - l_k)_+] + \int (u - l_k)_+^2 \right). \end{aligned}$$

Applying the energy estimate, we get

$$\begin{aligned} A_k^2 &\leq C \left( \int f(u - l_k)_+ + \int (u - l_k)_+^2 \right) \\ &\leq C \left( \|f\|_{L^q} \left( \int (u - l_k)_+^{q'} \right)^{1/q'} + \int (u - l_k)_+^2 \right) \\ &\leq C^k (\|f\|_{L^q} + 1) (A_{k-1}^{p_2/q'} + A_{k-1}^{p_1/q'}), \end{aligned}$$

with the last step by Chebyshev inequality. By the assumption on  $q$ , we see that both powers of  $A_{k-1}$  are strictly greater than 2. It follows that there is a  $\delta > 0$  such that  $A_k \rightarrow 0$  as desired.

In the other case, proceed analogously, using the energy estimate from Theorem 3.4 (as improved in Remark 3.3, and using the modified  $l_k = \sup_{\Omega^c} u_0 + 1 - 2^{-k}$ , which always stay above  $u_0$ ).  $\square$

A straightforward modification of this procedure gives existence of admissible solutions to the nonlinear problem with  $\mathbf{b} = Tu$ . We sketch the argument below.

**Lemma 5.4.** Let  $T$  be as in Theorem 4.7,  $s = \frac{1}{2}$ , and  $f \in L^1 \cap L^q$  for some  $q > s$ . Assume Condition 5.1. Then there exists a function  $u$  with finite energy satisfying (for every  $\phi \in C_c^\infty(\mathbb{R}^n)$ )

$$B_L[u, \phi] + B_N[u, \phi] = \int f\phi + \int u \langle Tu, \nabla \phi \rangle.$$

Moreover,  $u$  is admissible, and  $u \in L^\infty(\mathbb{R}^n)$ , with

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n, f, \Gamma, T).$$

*Proof.* We will show that the corresponding approximate problem admits a solution  $u^\delta$  satisfying the same energy estimates as in Lemma 5.2. In other words, we construct a  $u^\delta$  such that for each  $\phi \in C_c^\infty(\mathbb{R}^n)$  we have

$$B_L[u^\delta, \phi] + B_N[u^\delta, \phi] + \delta \int |\nabla u^\delta|^2 = \int f\phi + \int u^\delta \langle Tu^\delta, \nabla \phi \rangle,$$

which in addition satisfies

$$B_N[u^\delta, u^\delta] + B_N[u^\delta, u^\delta] \leq C(f).$$

We can then extract a subsequence  $u^{\delta_k} \rightarrow u$  in  $L^2_{\text{loc}}$ , with  $\nabla u^{\delta_k} \rightharpoonup \nabla u$  weakly in  $L^2$  and  $u^{\delta_k} \rightharpoonup u$  weakly in the  $H^s$  seminorm. Since  $\phi$  is compactly supported, we recover our problem (P) in the limit, as well as the energy estimate. The level set energy estimates from Lemma 5.2 are justified similarly, and applying the proof of Lemma 5.3 gives the final conclusion.

To find a solution to the approximate problem, set  $F(h, \delta, R)$  to be the solution of the problem

$$B_L[v, \phi] + B_N[v, \phi] + \delta \int |\nabla v|^2 = \int f\phi + \int v \langle Th, \nabla \phi \rangle$$

for all  $\phi \in C_c^\infty(B_R)$ , with  $v \equiv 0$  outside  $B_R$ . By Lemma 5.2, this exists for any  $h \in L^2(B_R)$  and satisfies

$$B_L[F(h, \delta), F(h, \delta)] + B_N[F(h, \delta), F(h, \delta)] \leq C(f).$$

This implies that the mapping  $h \rightarrow F(sh, \delta, R)$  is a compact map  $L^2(B_R) \rightarrow L^2(B_R)$  for each  $h, s$ . Applying the Leray-Schauder fixed point theorem shows there is a  $u_R^\delta$  satisfying  $u_R^\delta = F(u_R^\delta, \delta, R)$ . Now take  $R \rightarrow \infty$ , as in the proof of Lemma 5.2, to obtain a solution to (P $^\delta$ ) with appropriate energy estimates.  $\square$

We turn to the question of Hölder regularity for the Dirichlet problem for (P). The following theorem is basically an immediate consequence of Theorem 4.6, together with standard modifications near the boundary and the  $L^\infty$  bound above. We only sketch the proof.

**Theorem 5.5.** *Let  $u$  be an admissible solution of (P) on a bounded Lipschitz domain  $\Omega$ , with  $u_0$  bounded and  $f \in L^q$  for  $q > \frac{n}{2s}$ . Then:*

1. *If  $s \geq \frac{1}{2}$ ,  $\mathbf{b} \in L^q$  and  $q > \frac{n}{2s-1}$ , or if  $\mathbf{b} = 0$  and  $q > \frac{n}{s}$ , then there is an  $\alpha > 0$  such that for every  $\Omega' \subset\subset \Omega$  there is a constant  $C(\Omega')$  such that*

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq C (\|f\|_{L^q(\Omega)} + \|\mathbf{b}\|_{L^q(\Omega)} + \|u_0\|_{L^\infty}).$$

2. *If, in addition to the assumptions in (1), at  $x_0 \in \partial\Omega$ ,  $u_0$  satisfies  $|u_0(x_0) - u_0(y)| \leq C_0|x_0 - y|^{\alpha_0}$ , then there are  $\alpha, C$  depending on  $\alpha_0, C_0$  such that*

$$|u(x_0) - u(y)| \leq C (1 + \|f\|_{L^q(\Omega)} + \|\mathbf{b}\|_{L^q(\Omega)} + \|u_0\|_{L^\infty}) |x_0 - y|^\alpha.$$

3. If, in addition to the assumptions in (1),  $u_0 \in C^{0,\alpha_0}(\Omega^c)$ , then there are  $\alpha, C$  depending on  $\alpha_0$  such that

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C \left( \|f\|_{L^q(\Omega)} + \|\mathbf{b}\|_{L^q(\Omega)} + \|u_0\|_{C^{0,\alpha_0}(\Omega^c)} \right).$$

*Proof.* For (1), use Lemma 5.3 to estimate  $u \in L^\infty$ . After a dilation (which does not increase the norms of  $f, \mathbf{b}$ ), a cylinder  $A(E_1)$ , where  $A$  is an isometry, can be found so that  $A(E_1)$  is centered at any point  $x \in \Omega' \cap \Gamma$  and contained in  $\Omega$ . Theorem 4.6 then gives a uniform Hölder modulus of continuity at each  $x \in \Omega' \cap \Gamma$ . On the other hand, standard regularity theory (see [18, 6]) can be applied to give a modulus of continuity at every  $x \notin \Gamma$  of the form  $|u(x) - u(y)| \leq C|x - y|^\alpha d(x, \Gamma)^{-p}$  for some  $p$ . Combining these gives the conclusion.

For (2), if  $x_0$  is in  $\Gamma^c$  this is immediate from standard regularity theory, while if  $x_0 \in \Gamma$  a modification of the local estimate Theorem 4.6 can be applied; the modification is identical to the local case, as in [18]. Finally, (3) follows easily from (2).  $\square$

The same conclusions hold if  $\mathbf{b} = Tu$  and  $s = \frac{1}{2}$ ; simply apply Theorem 4.7 in place of Theorem 4.6. Next, we show how a solution can be localized on a cylinder about the origin.

**Lemma 5.6.** *Let  $u$  be an admissible solution to (P) on  $B_1$ . Let  $\eta \in C_c^\infty(E_2)$  be a smooth cutoff with  $\eta \equiv 1$  on  $E_{3/2}$ . Then  $\eta u$  is an admissible solution to (P) on  $E_1$  with the same drift and with right-hand side  $\tilde{f} = f + f_1$  satisfying the following estimate:*

$$\|f_1\|_{L^\infty(E_1)} \leq C(n, \lambda, \Lambda) \left\| \frac{u}{(1 + |y|)^{n+2s}} \right\|_{L^1(\mathbb{R}^n)}.$$

*Proof.* We perform the computation below on  $u$ ; an analogous argument for the approximations  $u^\delta$  would give the admissibility of  $\eta u$ . Let  $\psi \in C_c^\infty(E_1)$  and use  $\psi$  as a test function for  $u$ .

$$B_L[u, \psi] + B_N[u, \psi] = \int u \langle \mathbf{b}, \nabla \psi \rangle + f \psi.$$

The first term on the right can be rewritten as

$$\int u \langle \mathbf{b}, \nabla \psi \rangle = \int \eta u \langle \mathbf{b}, \nabla \psi \rangle$$

since  $\eta = 1$  on the support of  $\psi$ . Likewise, the local term on the left can be rewritten as

$$B_L[u, \psi] = B_L[\eta u, \psi].$$

For the nonlocal term, we must take care of some other quantities:

$$\begin{aligned} B_N[u, \psi] &= B_N[\eta u, \psi] + B_N[(1 - \eta)u, \psi] \\ &= B_N[\eta u, \psi] - \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)} \frac{a(x, y)[(1 - \eta)(x)u(x)\psi(y) + (1 - \eta)(y)u(y)\psi(x)]}{|x - y|^{n+2s}} dx dy \\ &= B_N[\eta u, \psi] - \int_{E_1} \psi(x) \int_{\mathbb{R}^n} \frac{[a(x, y) + a(y, x)]1_{\mathbb{R}^n \times \mathbb{R}^n \setminus (\Omega_1 \times \Omega_1)}(x, y)(1 - \eta)(y)u(y)}{|x - y|^{n+2s}} dy dx \\ &= B_N[\eta u, \psi] - \int_{E_1} \psi f_1. \end{aligned}$$



We thus have

$$B_N[\eta u, \psi] + B_L[\eta u, \psi] = \int \eta u \langle \mathbf{b}, \nabla \psi \rangle + (f + f_1) \psi,$$

and

$$\sup_{E_1} |f_1| \leq 2\Lambda \int_{E_{3/2}^c} \frac{|u(y)|}{|y|^{n+2s}} dy \leq C \left\| \frac{u}{(1+|y|)^{n+2s}} \right\|_{L^1(\mathbb{R}^n)},$$

completing the proof.  $\square$

*Remark 5.7.* If  $f$  was smooth,  $f_1$  will not be. However, smoothness within each of  $\Omega_1, \Omega_2$  will be preserved by this procedure. This is a feature of the way we chose to express the problem, as  $f$  serves as the right-hand side for two effectively different equations on the two domains.

Now we combine the previous lemma with a standard boundary flattening argument to relate the global solution  $u$  to a local solution in a regularized domain.

**Lemma 5.8.** *Let  $u$  be an admissible solution to (P) on  $E_1$ , and assume  $u \in H^1(\Omega_1 \cap E_2) \cap H^s(E_2)$ , as well as that  $\frac{u}{(1+|y|)^{n+2s}}$  is integrable. Then there is a bilipschitz transformation  $Q : E_2 \rightarrow \mathbb{R}^n$  and a function  $w : \mathbb{R}^n \rightarrow \mathbb{R}$  such that:*

- $E_r \subset Q(E_1) \subset E_{2r}$ .
- $Q(\Gamma \cap E_2) \subset \{x_n = 0\}$ .
- $w(Qx) = u(x)$  for every  $x \in E_1$ .
- $w$  solves (P) on  $E_r$  with data  $w_0, \bar{f}, \bar{\mathbf{b}}, \bar{A}, \bar{a}$ .
- There exist  $0 < \bar{\lambda} < \bar{\Lambda} < \infty$  depending on  $n, \lambda, \Lambda, Q$  such that  $\bar{\lambda}I \leq A \leq \bar{\Lambda}I$  and  $\bar{\lambda} \leq a \leq \bar{\Lambda}$ .
- $|w_0| \leq |u \circ Q^{-1}|$  on  $Q(E_2)$ .
- $\text{supp } w \subset Q(E_2)$ .
- $\|\bar{f}\|_{L^p} \leq C \left[ \|f\|_{L^p} + \left\| \frac{u}{(1+|y|)^{n+2s}} \right\|_{L^1} \right]$ .
- $\bar{\mathbf{b}} = (\nabla Q^T \mathbf{b}) \circ Q^{-1}$ .

*Proof.* Recall the standard assumption that  $\Gamma$  is given locally as a graph  $\{x_n = g(x')\}$ , with  $g$  Lipschitz. Set  $Q(x) = x - g(x')e_n$ ; then  $Q$  is Lipschitz on  $E_2$ ,  $Q(0) = 0$ ,  $Q(\Gamma \cap E_2) \subset \{x_n = 0\}$ ,  $\det \nabla Q = 1$ , and  $c_1 \leq |\nabla Q| \leq \frac{1}{c_1}$ . Fix a cutoff  $\eta \in C_c^\infty(E_2)$ , with  $\eta \equiv 1$  on  $E_{3/2}$ , and set

$$w(x) = \begin{cases} \eta(Qx)u(Qx) & x \in E_2 \\ 0 & x \notin E_2 \end{cases}.$$

From Lemma 5.6, we know that  $\eta u$  satisfies (P) with the same drift and a right-hand side  $f + f_1$ . We now compute the equation for  $w$ . Let  $\phi \in C_c^\infty(Q(E_1))$ , and use  $\phi \circ Q$  as a test function for  $\eta u$ :

$$B_L[\eta u, \phi \circ Q] + B_N[\eta u, \phi \circ Q] = \int_{E_1} \eta u \langle \mathbf{b}, \nabla \phi \circ Q \rangle + (f + f_1) \phi \circ Q.$$

The first term transforms in the standard way:

$$\begin{aligned}
B_L[\eta u, \phi \circ Q] &= \int \langle A(x) \nabla \eta u(x), \nabla Q \nabla \phi(Q(x)) \rangle dx \\
&= \int \langle A(Q^{-1}y) \nabla Q(Q^{-1}y) \nabla w(y), \nabla Q(Q^{-1}y) \nabla \phi(y) \rangle dy \\
&= \bar{B}_L[w, \phi],
\end{aligned}$$

where  $\bar{A} = (\nabla Q^T A \nabla Q) \circ Q^{-1}$ . For the nonlocal term, we can do a similar computation:

$$\begin{aligned}
B_N[\eta u, \phi \circ Q] &= \int_{E_2 \times E_2 \setminus (\Omega_1 \times \Omega_1)} \frac{[\eta u(x) - \eta u(y)] a(x, y) [\phi(Qx) - \phi(Qy)] dx dy}{|x - y|^{n+2s}} \\
&= \int_{Q(E_2) \times Q(E_2) \setminus (\Omega_1 \times \Omega_1)} \frac{[w(x') - w(y')] a(Q^{-1}x', Q^{-1}y') [\phi(x') - \phi(y')]}{|Q^{-1}x' - Q^{-1}y'|^{n+2s}} dx' dy' \\
&= \bar{B}_N[w, \phi]
\end{aligned}$$

where  $\bar{a}$  is given by

$$\bar{a}(x, y) = \begin{cases} a(Q^{-1}x, Q^{-1}y) \left( \frac{|x-y|}{|Q^{-1}x' - Q^{-1}y|} \right)^{n+2s} & x, y \in Q(E_2) \\ 1 & \text{otherwise} \end{cases}.$$

Note that as  $Q$  is bilipschitz, the quantity  $\left( \frac{|x-y|}{|Q^{-1}x' - Q^{-1}y|} \right)^{n+2s}$  is bounded above and below. Moreover, as both  $w$  and  $\phi$  are compactly supported on  $Q(E_2)$ ,  $\bar{a}$  can be extended arbitrarily for other  $x, y$  provided it stays elliptic and symmetric.

For the terms on the right-hand side, we get

$$\begin{aligned}
\int_{E_1} u(x) \langle b(x), \nabla Q(x) \nabla \phi(Qx) \rangle dx &= \int_{Q(E_1)} w(y) \langle \nabla Q^T(Q^{-1}y) b(Q^{-1}y), \nabla \phi(y) \rangle dy \\
&= \int_{Q(E_1)} w \langle \bar{b}, \nabla \phi \rangle
\end{aligned}$$

and

$$\int_{E_1} [f(x) + f_1(x)] \phi(Qx) = \int_{Q(E_1)} [f(Q^{-1}y) + f_1(Q^{-1}x)] \phi(y) dy = \int_{Q(E_1)} \bar{f} \phi.$$

The estimates then follow immediately.  $\square$

*Remark 5.9.* If  $\Gamma$  is smooth, it is simple to check that smoothness of the transformed functions is preserved (except  $f$ ; see the previous remark).

## 6 Parabolic Problem: Hölder Continuity

This section shows an elementary argument for deducing Hölder estimates for solutions of the natural parabolic analogue to  $(P)$  in the simple case that the coefficients are

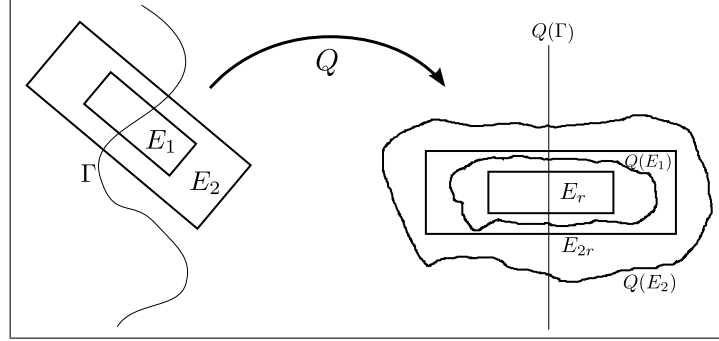


Figure 5.1: This is a diagram of the interface flattening. In fact, the flattening we consider preserves more structure than the illustration shows (for instance, the image of the top and bottom faces of  $E_1$  will still be graphs), but this will never be required.

independent of time. The more general case seems substantially more difficult, since the rescalings not only exhibit the increasing ellipticity ratio of  $(P_\epsilon)$  but also evolve at different time scales.

Set  $H = H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$  and  $H^*$  the dual space to  $H$ . We say  $u \in L^2([0, T]; H) \cap H^1([0, T]; H^*)$  is a weak solution of  $(P^*)$  on  $\Omega \times [0, T]$  if for a.e.  $t \in (0, T]$  and  $v \in C_c^\infty(\Omega)$  we have

$$\langle \partial_t u, v \rangle + B_L[u, v] + B_N[u, v] = \int_\Omega f v,$$

where  $\langle \cdot, \cdot \rangle$  denotes the  $H, H^*$  duality pairing. For the rest of this section, we ignore the issue of justifying energy inequalities; analogously to Section 3, weak solutions satisfying appropriate families of energy inequalities can be recovered from limits of, say, Galerkin approximations.

Our method is as follows: an argument analogous to that in Lemma 5.3 easily gives an  $L^\infty$  estimate. If the coefficients are independent of  $t$ , then the time derivative will satisfy a similar equation, and so will also be bounded. This means for each fixed time, the elliptic theory gives Hölder continuity of  $u$  in space, treating the time derivative term as part of the right-hand side.

**Lemma 6.1.** *Let  $u$  be a solution to  $(P^*)$  on  $\Omega \times [0, 2]$ ,  $f \in L^\infty$ , and assume that*

$$\sup_{\Omega^c \times [0, 2]} |u| \leq 1.$$

*Then  $u \in L^\infty(\Omega \times [1, 2])$ , and*

$$\sup_{\Omega \times [1, 2]} |u| \leq C \left( 1 + \|u\|_{L^2(\Omega \times [0, 2])} + \|f\|_{L^\infty} \right).$$

*Proof.* Assume first that  $\|f\|_{L^\infty} \leq 1$ . Set  $l_k = 2 - 2^{-k}$  and use  $(u - l_k)_+$  as a test function, observing it is supported on  $\Omega$ . This gives

$$\int_\Omega \partial_t u (u - l_k)_+ + B_L[u, (u - l_k)_+] + B_N[u, (u - l_k)_+] = \int_\Omega f (u - l_k)_+,$$

which easily yields

$$\sup_{[1-2^{-k-1}, 2]} \int_{\Omega} (u - l_k)_+^2 + \int_{1-2^{-k-1}}^2 \|(u - l_k)_+\|_H^2 \leq C^k \int_{1-2^{-k}}^2 (u - l_k)_+^2 + |f|(u - l_k)_+.$$

Set

$$A_k = \int_{1-2^{-k}}^2 \int_{\Omega} (u - l_k)_+^2 + (u - l_k)_+.$$

Then applying Chebyshev's inequality,

$$A_{k+1} \leq C^k \int_{1-2^{-k-1}}^2 \int_{\Omega} (u - l_k)_+^p$$

with  $p$  such that the inclusion  $L^p(\mathbb{R}^n \times \mathbb{R}) \subset L^\infty(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}; L^{p_1}(\mathbb{R}^n))$  is valid for  $p_1 = \frac{2n}{n-2s}$  (importantly,  $p > 2$ ). From Sobolev embedding and interpolation of Lebesgue spaces,

$$A_{k+1} \leq C^k \left( \|(u - l_k)_+\|_{L^\infty([1-2^{-k-1}, 2]; L^2(\Omega))}^2 + \|(u - l_k)_+\|_{L^2([1-2^{-k-1}, 2]; H)}^2 \right)^{p/2} \leq C^k A_k^{p/2}.$$

The final step used the energy estimate. Thus if  $A_0 \leq \delta$  for some small universal  $\delta$ ,  $A_k \rightarrow 0$  and

$$\sup_{\Omega \times [1, 2]} u \leq 2.$$

But we have that  $A_0 \leq \frac{\delta}{2} + C_\delta \|u\|_{L^2(\Omega \times [0, 2])}^2 < \delta$  if  $\|u\|_{L^2(\Omega)} < \sqrt{\delta/2C_\delta}$ . Apply this to  $\pm \sqrt{\delta/2C_\delta} u / (1 + \|u\|_{L^2} + \|f\|_{L^\infty})$  to get the desired result.  $\square$

**Lemma 6.2.** *Let  $u$  be a solution of  $(P^*)$  on  $\Omega \times [0, 3]$  and assume that none of  $A, a, f$  depend on  $t$ . Assume further that  $u = u_0$  on  $\Omega^c$ , with*

$$\left\| \sup_{t, s \in [0, 3]} \frac{|u_0(x, t) - u_0(x, s)|}{|x - t|} \right\|_{H^1(\Omega_1) \cap H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)} \leq 1.$$

Then  $\partial_t u$  is given by a bounded function, and satisfies the estimate

$$\sup_{\Omega \times [2, 3]} |\partial_t u| \leq C \left( 1 + \|u\|_{L^2(\Omega \times [0, 3])} + \|f\|_{L^2(\Omega \times [0, 3])} \right).$$

*Proof.* Use  $\partial_t(u - u_0)$  as a test function for  $(P^*)$ :

$$\begin{aligned} & \int_{\Omega} |\partial_t u|^2 + B_L[u, \partial_t u] + B_N[u, \partial_t u] \\ &= \int_{\Omega} f \partial_t(u - u_0) + B_L[u, \partial_t u_0] + B_N[u, \partial_t u_0] + \int_{\Omega} \partial_t u \partial_t u_0 \end{aligned}$$

and then reabsorb to get

$$\int_{\Omega} |\partial_t u|^2 + \partial_t B_L[u, u] + \partial_t B_N[u, u] \leq C \left( 1 + \int_{\Omega} f^2 + B_L[u, u] + B_N[u, u] \right).$$

Integrating in time gives the following:

$$\begin{aligned}
& \int_1^3 \int_{\Omega} |\partial_t u|^2 + \sup_{[1,3]} (B_L[u, u] + B_N[u, u]) \\
& \leq C \left[ 1 + \int_{\Omega \times [0,3]} f^2 + \int_{\frac{1}{2}}^3 B_L[u, u] + B_N[u, u] \right] \\
& \leq C \left[ 1 + \int_0^3 \int_{\Omega} f^2 + u^2 \right]
\end{aligned} \tag{6.1}$$

with the last step from an energy estimate as in the previous lemma. Now observe that  $\partial_t u$  solves  $(P^*)$  with 0 right-hand side and boundary data  $\partial_t u|_{\Omega^c}$ , which were assumed to be bounded. Thus Lemma 6.1 applies to give

$$\sup_{\Omega \times [2,3]} |\partial_t u| \leq C \left( 1 + \|\partial_t u\|_{L^2(\Omega \times [1,3])} \right) \leq C \left( 1 + \|f\|_{L^2(\Omega \times [0,3])} + \|u\|_{L^2(\Omega \times [0,3])} \right)$$

by combining with equation (6.1).  $\square$

*Remark 6.3.* This lemma can be iterated (provided the data is smooth) to obtain arbitrary regularity in  $t$ . At this point, Theorem 5.5 can be applied to obtain  $u \in C^{0,\alpha}(\Omega \times [\frac{5}{2}, 3])$ , and solutions will generally behave like the solutions to the elliptic problem. This argument also works with time-independent drift, or with coefficients varying smoothly with time.

## 7 Optimal Regularity for Simple Case

### 7.1 General Discussion, Tangential Regularity, and Reductions

In this section we begin with a discussion of a simplified situation, outlined as follows:

- Let  $\Gamma = \{x_n = 0\}$  be flat.
- Set  $\Omega = E_2 = \{|x'| < 2, |x_n| < 2\}$ .
- The coefficients are  $A = I$  and  $a = 1_{\Omega_2 \times \Omega_2} + \nu 1_{\mathbb{R}^n \setminus (\Omega_2 \times \Omega_2)}$  for a strictly positive  $\nu$ .
- The drift  $\mathbf{b} = 0$ .
- The right-hand side  $f$  is smooth in  $\Omega_1$  and  $\Omega_2$ , and also bounded.
- The data  $u_0$  is smooth and globally bounded.

We begin by deriving the classical form of the equation  $(P)$  satisfied by (an admissible)  $u$  on  $E_2$ . In this case, solutions correspond to minimizers of the strictly convex energy (1.6), and so are easily seen to be unique. First, take  $\psi \in C_c^\infty(E_2 \cap \Omega_1)$ . Then we obtain the following:

$$\begin{aligned}
& \int_{\Omega_1} f \psi \int_{\Omega_1} \langle \nabla \psi, \nabla u \rangle + \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[\psi(x) - \psi(y)]a(x, y)[u(x) - u(y)] dx dy}{|x - y|^{n+2s}} \\
& = - \int_{\Omega_1} u \Delta \psi + \nu \int_{\mathbb{R}^n} \psi(x) \int_{\Omega_2} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy dx.
\end{aligned}$$

As  $u$  is bounded from Lemma 5.3 (indeed, from Theorem 4.6, we know that  $u \in C^{0,\alpha}(\bar{E}_2)$ ), the integral on the right is bounded by  $Cd(\text{supp}\psi, \Gamma)^{-2s}$ . It follows that  $u$  is a distributional solution of

$$-\Delta u(x) = f(x) - \nu \int_{\Omega_2} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \quad \text{for } x \in E_2 \cap \Omega_1. \quad (7.1)$$

and so (7.1) is satisfied classically and  $u \in C^\infty(E_2 \cap \Omega_1)$ .

Alternatively, take a smooth  $\psi \in C_c^\infty(E_1 \cap \Omega_2)$  and use it as a test function:

$$\begin{aligned} \int_{\Omega_2} f\psi &= \int_{\Omega_2} \int_{\mathbb{R}^n} \frac{[\psi(x) - \psi(y)]a(x, y)[u(x) - u(y)]dx dy}{|x - y|^{n+2s}} \\ &= \int_{\Omega_2} \psi(x) \left[ 2 \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy - (2 - \nu) \int_{\Omega_1} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \right], \end{aligned}$$

meaning  $u$  solves

$$\frac{2}{c_s}(-\Delta)^s u(x) = f(x) + (2 - \nu) \int_{\Omega_1} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \quad (7.2)$$

distributionally on  $\Omega_2 \cap E_2$ . As on the other side, we have that  $u \in C^\infty(E_{3/2} \cap \Omega_2)$  and solves the equation classically.

Next, notice that for any unit vector  $e$  orthogonal to  $e_n$ ,  $\partial_e u$  solves a similar equation. Indeed, let  $\psi \in C_c^\infty(E_{3/2})$  and use  $\delta_{e,h}\psi(x) = \frac{u(x+he) - u(x)}{h}$  as a test function for  $|h| < \frac{1}{2}$ :

$$0 = B_L[u, \delta_{e,h}\psi] + B_N[u, \delta_{e,h}\psi] - \int f \delta_{e,h}\psi = B_L[\delta_{e,h}u, \psi] + B_N[\delta_{e,h}u, \psi] - \int \psi \delta_{e,h}f,$$

where we have used the translation-invariance of the forms  $B_L, B_N$ . We claim it follows that  $\delta_{e,h}u$  is uniformly bounded in  $H^1(\Omega_1 \cap E_1) \cap H^s(E_1)$  (this requires a standard argument with difference quotients outlined in the next section in a more general situation; see Lemma 8.2). This means that  $\partial_e u$  is in the same space and solves

$$B_L[\partial_e u, \psi] + B_N[\partial_e u, \psi] = \int \psi \partial_e f$$

in  $E_1$ . Applying Lemma 5.3 gives that  $\partial_e u$  is bounded, and indeed we know it is Hölder continuous. This can be iterated to show all higher tangential derivatives of  $u$  are in  $C^{0,\alpha}(\bar{E}_1)$ . In particular, the restriction  $u|_\Gamma \in C^\infty$ .

We now discuss the behavior in the normal direction  $e_n$ , which is not a trivial consequence of standard elliptic theory, like the above, and is more subtle. First, set

$$v(x', x_n) = u(x', 0),$$

and take any  $\psi \in C_c^\infty(E_1)$ . Then using the smoothness of  $v$ ,

$$\begin{aligned}
& B_L[v, \psi] + B_N[v, \psi] \\
&= \int_{\Omega_2 \times \mathbb{R}^n} \frac{[v(x) - v(y)]a(x, y)[\psi(x) - \psi(y)]}{|x - y|^{n+2s}} dx dy + \int_{\Omega_1} \sum_{i < n} \partial_i v \partial_i \psi \\
&= \int \psi(x) \left[ \int_{\Omega_2} \frac{[v(x) - v(y)]a(x, y)dy}{|x - y|^{n+2s}} + 1_{\Omega_2}(x) \int_{\mathbb{R}^n} \frac{[v(x) - v(y)]a(x, y)dy}{|x - y|^{n+2s}} \right] dx \\
&\quad + \int_{\Omega_1} \psi(-\Delta v).
\end{aligned}$$

But as  $v$  doesn't depend on  $x_n$ , the integrand is bounded (when interpreted in the principal value sense):

$$\left| \int_{\Omega_2} \frac{[v(x) - v(y)]a(x, y)dy}{|x - y|^{n+2s}} \right| \leq C \|v\|_{C^2 \cap L^\infty},$$

while the other term is just the fractional Laplacian. Thus

$$B_L[v, \psi] + B_N[v, \psi] = \int \tilde{f} \psi,$$

with  $\tilde{f}$  bounded. It can be verified that  $\tilde{f}$  is piecewise smooth as well. It follows that  $u - v$  satisfies the same hypotheses as  $u$  and vanishes on  $\Gamma$ . We will therefore restrict the discussion from here on to  $u$  vanishing along  $\Gamma$ . A solution  $u$  satisfying all of the above will be said to satisfy (H). More precisely:

**Definition 7.1.** A solution to (P) is said to *satisfy (H)* if:

1.  $u$  solves (P) in  $E_2$
2.  $u \in C^{0, \alpha_*}(\mathbb{R}^n) \cap H^s(E_4) \cap H^1(E_4 \cap \Omega_1)$
3.  $u \in C^\infty(\Omega_1 \cap E_2) \cap C^\infty(\Omega_2 \cap E_2)$
4. For any  $\{e^i\}_{i=1}^k$  with each  $e^i \perp e_n$ ,  $\partial_{e^1} \partial_{e^2} \cdots \partial_{e^k} u \in C^{0, \alpha_*}(E_2)$
5.  $u = 0$  on  $\Gamma$
6.  $f \in C^\infty(\Omega_2 \cap E_2) \cap C^\infty(\Omega_1 \cap E_2) \cap L^\infty(E_2)$

Here we have changed the size of the cylinders for convenience. We will now give a formal construction of a function  $\Phi$  which vanishes on  $\Gamma$  and is a solution to (7.1) and (7.2) simultaneously (for some  $f$ ).

Let

$$A(n, s) = \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy'.$$

Consider homogeneous functions of one variable of the form

$$\rho_\alpha(t) = 1_{\{t \geq 0\}} t^\alpha.$$

Then for  $t > 0$ ,

$$\int_{\mathbb{R}} \frac{\rho_\alpha(t) - \rho_\alpha(r)}{|t-r|^{1+2s}} dr = t^{\alpha-2s} \left[ \int_0^\infty \frac{1-r^\alpha}{|1-r|^{1+2s}} dr + \int_0^\infty \frac{1}{(1+r)^{1+2s}} dr \right] = t^{\alpha-2s} q(s, \alpha)$$

is homogeneous of degree  $\alpha - 2s$  (where the integral is interpreted in the principal value sense).

**Claim 7.2.** For  $s$  fixed,  $q$  is continuous and concave in  $\alpha \in (0, 2s)$ , with

$$\lim_{\alpha \rightarrow 0^+} q(s, \alpha) = \frac{1}{2s}$$

$$\lim_{\alpha \rightarrow 2s^-} q(s, \alpha) = -\infty.$$

If  $s \leq \frac{1}{2}$ , then  $q(s, \cdot)$  is decreasing. If  $s \geq \frac{1}{2}$ , then  $q$  is symmetric about  $\alpha = s - \frac{1}{2}$ ; in particular,  $q(s, 2s-1) = \frac{1}{2s}$  and  $q$  is decreasing on  $(2s-1, 2s)$ . In addition,  $q(s, 1) = \frac{1}{2s} + \frac{1}{1-2s}$ .

*Proof.* In [3], the integral defining  $q(s, \alpha)$  is shown to admit the following simplified form:

$$q(s, \alpha) = \frac{1}{2s} - \int_0^1 \frac{(t^\alpha - 1)(1 - t^{2s-1-\alpha})}{(1-t)^{1+2s}} dt.$$

The integrand is symmetric across  $\alpha = s - \frac{1}{2}$  for each  $t$  and convex, so  $q$  is symmetric and concave. It is clear from this expression that  $q(s, 0^+) = q(s, 2s-1) = \frac{1}{2s}$ , and  $q$  is decreasing for  $s > (2s-1)_+$ . The limit as  $\alpha \rightarrow 2s$  follows from monotone convergence theorem. The special value  $q(s, 1)$  can be obtained by applying elementary integration techniques. See also [4] for the value of  $q(s, s - \frac{1}{2})$  and additional properties.  $\square$

We return to the construction. Fix  $\chi(t)$  a cutoff compactly supported on  $[-2, 2]$  and identically 1 on  $[-1, 1]$ . Then find  $\alpha_0 \in ((2s-1)_+, 2s)$  such that  $q(s, \alpha_0) = (1 - \frac{\nu}{2}) \frac{1}{2s} < \frac{1}{2s}$ . Set  $v_0(x', x_n) = \rho_{\alpha_0}(x_n)$ . Then for  $x \in \Omega_2$ ,

$$\begin{aligned} \frac{1}{c_s} (-\Delta)^s v_0(x) &= \int_{\mathbb{R}^n} \frac{\rho_{\alpha_0}(x_n) - \rho_{\alpha_0}(y_n)}{|x-y|^{n+2s}} dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{\rho_{\alpha_0}(x_n) - \rho_{\alpha_0}(y_n)}{(|x_n - y_n|^2 + |x' - y'|^2)^{\frac{n+2s}{2}}} dy' dy_n \\ &= \int_{\mathbb{R}} \frac{\rho_{\alpha_0}(x_n) - \rho_{\alpha_0}(y_n)}{|x_n - y_n|^{1+2s}} dy_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \\ &= q(s, \alpha_0) A(n, s) |x_n|^{\alpha_0-2s} = (1 - \frac{\nu}{2}) \frac{1}{2s} A(n, s) |x_n|^{\alpha_0-2s}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{v_0(x)}{|x-y|^{n+2s}} dy &= (1 - \frac{\nu}{2}) |x_n|^{\alpha_0-2s} \int_{\mathbb{R}^n} \frac{1}{((|y_n|+1)^2 + |y'|^2)^{\frac{n+2s}{2}}} dy \\ &= (1 - \frac{\nu}{2}) A(n, s) |x_n|^{\alpha_0-2s} \int_0^\infty \frac{1}{(t+1)^{1+2s}} dt \\ &= (1 - \frac{\nu}{2}) A(n, s) \frac{1}{2s} |x_n|^{\alpha_0-2s} = \frac{1}{c_s} (-\Delta)^s v_0(x). \end{aligned}$$



Thus  $v_0$  satisfies (7.2) with 0 right-hand side. Let  $\tilde{v}_0(x', x_n) = \chi(x_n)\rho_{\alpha_0+(2-2s)}(-x_n)$  (note that  $2 + \alpha_0 - 2s < 2$ ). Then for  $x \in \Omega_1$ ,

$$-\Delta(v_0 + M_0\tilde{v}_0)(x) = -M_0(2 + \alpha_0 - 2s)(1 + \alpha_0 - 2s)\chi(x_n)|x_n|^{\alpha_0-2s} + C,$$

while

$$\begin{aligned} & -\nu \int_{\Omega_2} \frac{[v_0 + M_0\tilde{v}_0](x) - [v_0 + M_0\tilde{v}_0](y)}{|x - y|^{n+2s}} dy \\ & = -\nu \left[ M_0 A(n, s) \chi(x_n) |x_n|^{2+\alpha_0-4s} - A(n, s) \int_0^\infty \frac{t^{\alpha_0}}{(1+t)^{1+2s}} dt |x_n|^{\alpha_0-2s} \right] \end{aligned}$$

Since  $\alpha_0 > 1 - 2s$ ,  $1 + \alpha_0 - 2s \neq 0$ , so set

$$M_0 = -\frac{\nu}{(2 + \alpha_0 - 2s)(1 + \alpha_0 - 2s)} A(n, s) \int_0^\infty \frac{t^{\alpha_0}}{(1+t)^{1+2s}} dt.$$

Then  $\Phi_0 = v_0 + M_0\tilde{v}_0$  satisfies (7.1) with right-hand side homogeneous of degree  $\alpha_0 - 2s + (2 - 2s)$ . If this is nonnegative, the right-hand side is bounded, as is the right-hand side in (7.2), and the construction is complete. If not, we can add additional terms to reduce the homogeneity of the right-hand sides, as shown below.

Set  $v_1 = L_1\rho_{\alpha_0+(2-2s)}(x_n)$ , so that

$$\begin{aligned} & \frac{1}{c_s} (-\Delta)^s(\Phi_0 + v_1)(x) - (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{(\Phi_0 + v_1)(x) - (\Phi_0 + v_1)(y)}{|x - y|^{n+2s}} dy \\ & = |x_n|^{\alpha_0-2s+(2-2s)} A(n, s) L_1 \left[ q(s, \alpha_0 + (2 - 2s)) - (1 - \frac{\nu}{2}) \frac{1}{2s} \right] \\ & \quad - \frac{\nu}{2} M_0 A(n, s) \int_0^\infty \frac{\chi(t) t^{\alpha_0}}{(1+t)^{1+2s}} dt |x_n|^{\alpha_0-2s+(2-2s)}. \end{aligned}$$

As  $q(s, \alpha_0 + (2 - 2s)) < (1 - \frac{\nu}{2}) \frac{1}{2s} = q(s, \alpha_0)$ ,  $L_1$  can be chosen so this is 0. Now if  $\tilde{v}_1 = \chi(x_n) M_1 \rho_{\alpha_0+2(2-2s)}(-x_n)$ , a computation as before will reveal that if  $\alpha_0 + 2(2 - 2s) \neq 1$  (which is always satisfied), some choice of  $M_1$  gives that  $\Phi_1 = \Phi_0 + v_1 + \tilde{v}_1$  satisfies (7.1) with right-hand side homogeneous of degree  $\alpha_0 - 2s + 2(2 - 2s)$  (plus a bounded term).

Now set  $k_*$  to be the smallest integer such that  $\alpha_0 - 2s + (k_* + 1)(2 - 2s) \geq 0$  and continue this procedure until  $\Phi := \chi(x_n)\Phi_k$  has been constructed. Then  $\Phi$  solves (7.1) and (7.2) with (locally) bounded right-hand side, vanishes on  $\Gamma$ , and to leading order behaves like  $1_{\{x_n > 0\}}|x_n|^{\alpha_0} + M_0 1_{\{x_n < 0\}}|x_n|^{\alpha_0+(2-2s)}$ . We will justify below that this function actually solves (P) in the weak sense. From the construction this is clear for test functions compactly supported on either  $\Omega_1$  or  $\Omega_2$ , but we still need to check what happens when the test function is supported near  $\Gamma$ . In the classical transmission problem, this kind of test function is related to the transmission condition.

**Proposition 7.3.** *The function  $\Phi$  solves (P) on  $E_1$  with bounded right-hand side.*

*Proof.* First, note that by the assumption on  $\alpha_0$ ,  $\Phi$  is in the energy space  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n)$ . Take any  $w \in C_c^\infty(E_1)$  and let  $\tau_\delta(t) = (1 - |t|/\delta)_+$ . Then we easily have that

$$B_L[\Phi, w(1 - \tau_\delta)] + B_N[\Phi, w(1 - \tau_\delta)] = \int Fw(1 - \tau_\delta)$$

for some bounded  $F$ . We show that

$$B_L[\Phi, w\tau_\delta] + B_N[\Phi, w\tau_\delta] \rightarrow 0$$

as  $\delta \rightarrow 0$ ; this will prove the proposition.

To see this, we compute the two forms directly:

$$\begin{aligned} B_L[\Phi, w\tau_\delta] &= -\frac{1}{\delta} \sum_{k=0}^{k_*} M_k \int_{-\delta}^0 \int_{\mathbb{R}^{n-1}} w(x', x_n) \frac{|x_n|^{-1+\alpha_0+(k+1)(2-2s)}}{\alpha_0 + (k+1)(2-2s)} + o_\delta(1) \\ &= -\int_{\mathbb{R}^{n-1}} w(x', 0) \sum_{k=0}^{k_*} M_k \delta^{-1+\alpha_0+(k+1)(2-2s)} + o_\delta(1) \\ &= o_\delta(1), \end{aligned}$$

because  $\alpha_0 + 2 - 2s > 2 - 2s + 2s - 1 > 1$ . On the other hand,

$$\begin{aligned} |B_N[\Phi, w\tau_\delta]| &\leq o_\delta(1) + C \sum_{k=0}^{k_*} \left| \int_{\Omega_2} \int_{\Omega_2} \frac{(|x_n|^{\alpha_0+2k(1-s)} - |y_n|^{\alpha_0+2k(1-s)}) (w\tau_\delta(x) - w\tau_\delta(y))}{|x-y|^{n+2s}} dx dy \right| \\ &\quad + \nu \left| \int_{\Omega_1} \int_{\Omega_2} \frac{(L_k |x_n|^{\alpha_0+2k(1-s)} - M_k |y_n|^{\alpha_0+2(k+1)(1-s)}) (w\tau_\delta(x) - w\tau_\delta(y))}{|x-y|^{n+2s}} dx dy \right|. \end{aligned}$$

The various terms work similarly; for example:

$$\begin{aligned} &\int_{\Omega_2} \int_{\Omega_2} \frac{(|x_n|^{\alpha_0+k(2-2s)} - |y_n|^{\alpha_0+k(2-2s)}) (w\tau_\delta(x) - w\tau_\delta(y))}{|x-y|^{n+2s}} dx dy \\ &= 2A(n, s) \int_{\mathbb{R}^{n-1}} w dx' \int_0^\infty \int_0^\infty \frac{t^{\alpha_0+k(2-2s)} (\tau_\delta(t) - \tau_\delta(s))}{|t-s|^{1+2s}} ds dt + o_\delta(1) \\ &= 2CA(n, s) \int_{\mathbb{R}^{n-1}} w dx' \delta^{-1+\alpha_0+(k+1)(2-2s)} + o_\delta(1) = o_\delta(1), \end{aligned}$$

where the first step integrated in  $x', y'$  and the second scaled out the  $\delta$  dependence. The other terms can be dealt with the same way.  $\square$

We note that the  $\alpha_0$  chosen in the construction is unique in the sense that no other value  $\alpha \in (0, 2s)$  would lead to an  $x_n$ -homogeneous solution of (7.2). More complex behavior may occur in the region of  $\nu \leq 0$  and  $s > \frac{1}{2}$ , where two distinct simultaneous solutions to (7.2) and (7.1) can be constructed, exactly one of which will have finite energy. These, however, do not correspond to  $(P)$  in the form we are considering.

## 7.2 Bootstrap Machinery, Near-Optimal Estimates

The following two lemmas work in parallel to bootstrap regularity for solutions  $u$  satisfying  $(H)$ .

**Lemma 7.4.** *Let  $u$  be a solution satisfying (H) and  $r > 1$ . Then we have:*

1. *If  $u \in C^{0,\alpha}(E_r)$  then  $u \in C^{0,(\alpha+2-2s)\wedge 1}(E_1 \cap \bar{\Omega}_1)$ .*
2. *If  $u \in C^{0,\alpha}(E_r)$ ,  $2 > \alpha + 2 - 2s > 1$ , and in addition  $\partial_{e_n} u(x', 0^-) = 0$  (from  $\Omega_1$ , in the sense of distributions), then  $u \in C^{1,\alpha+1-2s}(E_1 \cap \bar{\Omega}_1)$ .*
3. *If  $u \in C^{1,\alpha}(E_r)$  and  $\partial_{e_n} u(x', 0) = 0$ , then  $u \in C^{1,\alpha'+2-2s}(E_1 \cap \bar{\Omega}_1)$  for all  $\alpha' \leq \alpha$  with  $\alpha' < 2s - 1$ .*

*Proof.* Recall that  $u$  solves

$$-\Delta u(x) = f(x) - \nu \int_{\Omega_2} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy := g_1(x)$$

for  $x \in \Omega_1 \cap E_2$ . Our method will be to scale the following basic estimate for solutions of Laplace equation: if  $v$  solves

$$\begin{cases} -\Delta v = h & x \in B_1 \\ v = v_0 & x \in \partial B_1 \end{cases}$$

with  $h, v_0$  bounded, then for each  $\gamma < 1$  we have

$$\|v\|_{C^{1,\gamma}(B_{1/2})} \leq C_\gamma [\|v_0\|_{L^\infty(\partial B_1)} + \|h\|_{L^\infty(B_1)}].$$

Now, from (H) we have that  $\|u_{ee}\|_{L^\infty(E_2)} \leq C$  for each  $e \perp e_n$ . Thus

$$-u_{e_n e_n}(x) = \Delta_{\mathbb{R}^{n-1}} u(x) + f(x) - \nu \int_{\Omega_2} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy := g(x).$$

For (1) we have that  $|g(x', x_n)| \leq C(1 + |x_n|^{\alpha-2s})$  and  $u(x', 0) = 0$ , giving

$$|u_{e_n}(x', x_n)| \leq |u_{e_n}(x', -1)| + C \int_{-1}^{x_n} t^{\alpha-2s} dt \leq C(1 + |x_n|^{\alpha+1-2s}).$$

Another application of the fundamental theorem of calculus gives

$$|u(x', t) - u(x', s)| \leq C(|t - s| + |t - s|^{\alpha+2-2s}).$$

From this and tangential regularity we may easily deduce the desired estimate.

For (2), we have from (1) that  $\nabla u$  is bounded. Moreover,  $|g(x', x_n)| \leq C(1 + |x_n|^{\alpha-2s})$ , so

$$|u_{e_n}(x', t) - u_{e_n}(x', t')| \leq C \int_{t'}^t s^{\alpha-2s} ds \leq C|t - t'|^{\alpha+1-2s}.$$

In particular,  $\partial_{e_n} u$  extends continuously to  $\Gamma$ . But since  $\partial_{e_n} u(x', 0^-)$  is assumed to be 0 (in the distributional sense), we may further obtain that

$$|u(x', t)| \leq C t^{\alpha+2-2s}.$$

Together with the fact that  $|\Delta u(x', x_n)| \leq C(1 + |x_n|^{\alpha-2s})$ , we apply the estimate above to the function  $u_x(y) = r^{-\beta} u(r(y - x))$ , where  $r = \frac{|x_n|}{4}$  and  $\beta = \alpha + 2 - 2s$ .

This function solves Laplace equation on  $B_1$  with right-hand side  $f_x$  satisfying  $|f_x| \leq Cr^{-\beta+2+\alpha-2s} \leq C$ , and also  $|u_x| \leq Cr^{-\beta}r^{\alpha+2-2s} \leq C$ . Then applying the estimate we see that  $[u_x]_{C^\beta(B_{1/2})} \leq C$ , and so scaling back gives

$$|\nabla u(x) - \nabla u(y)| \leq C|x - y|^{\beta-1}$$

for every  $|x - y| \leq \frac{|x_n|}{8}$ . We claim this implies the conclusion, for

$$\begin{aligned} |\nabla u(x', t)| &= |\nabla u(x', t) - \nabla u(x', 0)| \\ &\leq \sum_{j=0}^{\infty} \left| \nabla u \left( x', \left( \frac{7}{8} \right)^j t \right) - \nabla u \left( x', \left( \frac{7}{8} \right)^{j+1} t \right) \right| \\ &\leq Ct^{\beta-1} \sum_{j=0}^{\infty} \left( \frac{1}{8} \right)^{j(\beta-1)} \\ &\leq Ct^{\beta-1}. \end{aligned}$$

Then when  $|x - y| \geq \max\{\frac{|x_n|}{8}, \frac{|y_n|}{8}\}$  use this to obtain

$$|\nabla u(x', x_n) - \nabla u(y', y_n)| \leq C(|x_n|^{\beta-1} + |y_n|^{\beta-1}) \leq C|x - y|^{\beta-1}.$$

For (3), proceed the same way, using the following improved estimate on  $g$ :

$$\begin{aligned} \left| \int_{\Omega_2} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \right| &\leq C|x_n|^{1+\alpha}|x_n|^{-2s} + \int_{\Omega_2} \frac{|y_n|^{1+\alpha}}{|x - y|^{n+2s}} dy \\ &\leq C|x_n|^{1+\alpha-2s}. \end{aligned}$$

□

**Lemma 7.5.** *Let  $u$  be a solution satisfying (H) and  $r > 1$ . Then we have:*

1. *If  $u \in C^{0,\alpha}(E_r \cap \bar{\Omega}_1)$ ,  $\alpha' < \min\{\alpha, \alpha_0\}$ , then  $u \in C^{0,\alpha'}(E_1 \cap \bar{\Omega}_2)$ .*
2. *If  $u \in C^{1,\alpha}(E_r \cap \bar{\Omega}_1)$ ,  $\alpha_0 > 1$ , and  $\partial_{e_n} u(x', 0^-) = 0$  (from  $\Omega_1$ ), then  $u \in C^{1,\alpha'}(E_1 \cap \bar{\Omega}_2)$  for each  $\alpha' < \min\{\alpha, \alpha_0 - 1\}$ , and  $\partial_{e_n} u(x', 0) = 0$ .*

*Proof.* The general idea will be to use the functions  $\rho_\alpha$  as barriers. There are, however, some technical issues, as  $\alpha$  may be small enough that  $\rho_\alpha$  does not lie in the energy space.

Fix  $\xi : \mathbb{R} \rightarrow [0, 1]$  a smooth cutoff such that  $\xi = 0$  on  $[-1, 1]$ ,  $\xi = 1$  on  $[-r, r]^c$ , and  $|(-\Delta)^s \xi| \leq C$ . Set  $w(x) = u(x) + C_1 \rho_{\alpha'}(x_n) + C_2 \rho_\alpha(-x_n) + C_3 \xi(|x'|)$ .  $w$  is a solution of

$$\begin{cases} \frac{1}{c_s} (-\Delta)^s w(x) = (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy + \frac{C_1}{c_s} (-\Delta)^s \rho_{\alpha'}(x_n) \\ \quad + C_2 \int_{\Omega_1} \frac{-|y_n|^\alpha}{|x - y|^{n+2s}} dy + C_3 (-\Delta)^s \xi \\ w(x) = C_1 \rho_{\alpha'}(x_n) + C_2 \rho_\alpha(-x_n) + C_3 \xi(|x'|) + u(x) \end{cases} \quad \begin{matrix} x \in E_r \cap \Omega_2 \\ x \in (E_r \cap \Omega_2)^c \end{matrix} \quad (7.3)$$

By choosing  $C_2, C_3$  large and using that  $u|_\Gamma = 0$  and  $u \in C^{0,\alpha}(E_r \cap \Omega_1)$ , we can arrange so that  $w \geq 0$  on  $(E_r \cap \Omega_2)^c$ . Rewriting the right-hand side,

$$\begin{aligned} \frac{1}{c_s}(-\Delta)^s w(x) &= (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{w(x) - w(y)}{|x - y|^{n+2s}} dy - C_1(1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{\rho_{\alpha'}(x_n) - \rho_{\alpha'}(y_n)}{|x - y|^{n+2s}} dy \\ &\quad + C_1 q(s, \alpha') A(n, s) |x_n|^{\alpha' - 2s} - CC_2 |x_n|^{\alpha - 2s} - CC_3 \\ &\geq (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{w(x) - w(y)}{|x - y|^{n+2s}} dy - CC_2 |x_n|^{\alpha - 2s} - CC_3 \\ &\quad + C_1 A(n, s) [q(s, \alpha') - q(s, \alpha_0)] |x_n|^{\alpha' - 2s} \\ &\geq (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{w(x) - w(y)}{|x - y|^{n+2s}} dy + |x_n|^{\alpha' - 2s} \end{aligned}$$

where the final step used that  $\alpha' < \alpha$  and that  $\alpha' < \alpha_0$  (the latter implying that  $q(s, \alpha') > q(s, \alpha_0)$ ), and  $C_1$  was chosen to be sufficiently large relative to  $C_2$  and  $C_3$ . Let  $m = \min_{E_r \cap \Omega_2} w$ ; we claim  $m = 0$ . Indeed, assume for contradiction that  $m < 0$ . Then the function  $(w - \frac{m}{2})_-$  is compactly supported on  $E_r \cap \Omega_2$  (here we use that  $w$  is continuous and nonnegative on  $\partial(E_r \cap \Omega_2)$ ). Then by multiplying (7.3) by  $(w - \frac{m}{2})_-$  and undoing the computation used to derive (7.2), we see that

$$B_N[w, (w - \frac{m}{2})_-] \leq \int |x_n|^{\alpha' - 2s} (w - \frac{m}{2})_-$$

since  $(w - \frac{m}{2})_-$  is negative. Note carefully that the left-hand side is finite despite  $w$  possibly having infinite  $H^s$  seminorm, since  $(w - \frac{m}{2})_-$  is supported away from  $\Gamma$ . Also, it is positive:

$$B_N[w, (w - \frac{m}{2})_-] = B_N[(w - \frac{m}{2})_-, (w - \frac{m}{2})_-] + B_N[(w - \frac{m}{2})_+, (w - \frac{m}{2})_-].$$

The first term is clearly positive, while the second is nonnegative from the same computation as in, say, (3.6). On the other hand, by definition of  $m$ , the right-hand side is strictly negative, giving a contradiction.

Applying this to  $\pm u$  gives that  $|u(x)| \leq C_2 |x_n|^{\alpha'}$  on  $E_1 \cap \Omega_2$ . Combining this with the already known regularity of  $u$  away from  $\Gamma$  gives (1). The same argument works for (2), giving  $|u(x)| \leq C_2 |x_n|^{1+\alpha'}$  on  $E_{\frac{1+r}{2}} \cap \Omega_2$ . Use this to estimate the right-hand side in (7.2) by  $|x_n|^{1+\alpha' - 2s}$ . Then argue as in the proof of Lemma 7.4, giving that  $u \in C^{1,\alpha'}(E_1 \cap \bar{\Omega}_2)$ ; we omit the details.  $\square$

**Lemma 7.6.** *Let  $u$  be a solution satisfying (H). Then if  $u \in C^{0,\alpha}(E_r \cap \bar{\Omega}_2)$  for some  $\alpha > (2s - 1)_+$  and  $r > 1$ ,  $\partial_{e_n} u = 0$  from  $\Omega_1$  for  $|x'| < 1$  (in the sense of distributions).*

*Proof.* Note that an application of Lemma 7.4 guarantees that  $u \in C^{0,\alpha}(E_{\frac{1+r}{2}})$ . We will show that

$$\int_{\Omega_1} \langle \nabla u, T \rangle = - \int_{\Omega_1} u \operatorname{div} T$$

for any vector field  $T \in C_c^\infty(E_{\frac{3+r}{4}})$ . It suffices to consider  $T(x) = w(x)e_n$ ; for any  $T \perp e_n$  the above is immediate from integrating by parts in tangential directions.

Set  $\tau_\delta(t) = (1 - \frac{|t|}{\delta})_+$ ; then

$$\begin{aligned} \int_{\Omega_1} w \partial_{e_n} u &= \lim_{\delta \rightarrow 0} \int_{\Omega_1} (1 - \tau_\delta(x_n)) w(x) \partial_{e_n} u(x) \\ &= \lim_{\delta \rightarrow 0} - \int_{\Omega_1} u(x) (1 - \tau_\delta(x_n)) \partial_{e_n} w(x) + \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{\mathbb{R}^{n-1}} \int_{-\delta}^0 u(x) w(x) \\ &= - \int_{\Omega_1} u \partial_{e_n} w + \lim_{\delta \rightarrow 0} \frac{1}{\delta} \int_{\mathbb{R}^{n-1}} \int_{-\delta}^0 u(x) w(x) \end{aligned}$$

provided the limit exists. We show that it does, and in fact equals 0, which immediately implies the conclusion. To this end, use  $w_\delta(x) = w(x) \tau_\delta(x_n)$  as a test function in (P). Then

$$B_L[u, w_\delta] = \int_{-\delta}^0 \int_{\mathbb{R}^{n-1}} w(x) \partial_{e_n} u(x', x_n) \frac{1}{\delta} dx' dx_n + o_\delta(t)$$

so it would suffice to show  $B_L[u, w_\delta] \rightarrow 0$ . From the equation,

$$B_L[u, w_\delta] + B_N[u, w_\delta] = \int f w_\delta \rightarrow 0$$

as  $\delta \rightarrow 0$ , so it will be enough to show  $B_N[u, w_\delta] \rightarrow 0$  instead. For this,

$$\begin{aligned} |B_N[u, w_\delta]| &= \left| \int_{\Omega_2 \times \mathbb{R}^n} \frac{[u(x) - u(y)] a(x, y) [w_\delta(x) - w_\delta(y)]}{|x - y|^{n+2s}} dx dy \right| \\ &\leq C(r) \int_{\Omega_2 \times \mathbb{R}^n} \frac{|x - y|^\alpha |w_\delta(x) - w_\delta(y)|}{|x - y|^{n+2s}} dx dy \\ &\leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|w(x) - w(y)| \tau_\delta(x_n) + |w(y)| |\tau_\delta(x_n) - \tau_\delta(y_n)|}{|x - y|^{n+2s-\alpha}} dx dy \\ &\leq C \int_{\mathbb{R}^n} \frac{\tau_\delta(x_n)}{1 + |y|^{n+2s-\alpha}} \\ &\quad + C \int_{\mathbb{R} \times \mathbb{R}} \frac{|\tau_\delta(t) - \tau_\delta(s)|}{|t - s|^{1+2s-\alpha}} dt ds \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{\sup_t |w(y', t)|}{(1 + |x' - y'|^2)^{\frac{n+2s-\alpha}{2}}} dx' dy' \\ &\leq o_\delta(1) + C \delta^{1-2s+\alpha} \int_{\mathbb{R}} \frac{1}{1 + t^{1+2s-\alpha}} dt = o_\delta(1). \end{aligned}$$

This proves the lemma.  $\square$

Combining these statements easily implies the following:

**Theorem 7.7.** *Let  $u$  satisfy (H). Then  $u \in C^{0,\alpha}(E_1 \cap \bar{\Omega}_2)$  for every  $\alpha < \alpha_0$ ,  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_1)$  for every  $\alpha < \alpha_0 + 1 - 2s$ , and  $\partial_{e_n} u(x', 0^-) = 0$  for  $|x'| < 1$ . Moreover, if  $\alpha_0 > 1$ , then  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_2)$  for each  $\alpha < \alpha_0 - 1$  and  $\partial_{e_n} u(x', 0) = 0$  (from both sides).*

*Proof.* From (H) we have  $u \in C^{0,\alpha_*}(\mathbb{R}^n)$  for some  $\alpha_*$ . If  $\alpha_* < 2s - 1$ , apply Lemma 7.4 to obtain  $u \in C^{0,\alpha_1}(E_{3/2} \cap \bar{\Omega}_1)$  for  $\alpha_1 = 2 - 2s + \alpha_*$ , and then Lemma 7.5 to get  $u \in C^{0,\alpha_2}(E_{1+2^{-2}})$ . Continue iterating this until  $u \in C^{0,\alpha_k}(E_{1+2^{-k}})$  for some  $\alpha_k > 2s - 1$ .

At that point apply 7.6, and continue iterating until the conclusion is reached. Since it can be arranged that the exponent improves by at least, say,  $1 - s$  every cycle, this will conclude in finitely many steps.  $\square$

This theorem shows almost-optimal regularity for solutions of  $(P)$ , at least in the simple case treated here. The next section will discuss to what extent this can be generalized to equations with translation invariance, while in the subsequent section we will study the case of variable coefficients.

### 7.3 Refined Barriers, Optimal Regularity, and Transmission Condition

In the previous subsection, we showed solutions come close to having the optimal Hölder exponent (relative to the special solution  $\Phi$ ). However, as we saw in the construction of that solution, there is reason to expect that solutions actually have the asymptotic behavior

$$u(x) = u(x', 0) + m\rho_{\alpha_0}(x_n) + mM_0\rho_{\alpha_0+2-2s}(-x_n) + o((x_n)_+^{\alpha_0}) + o(-(x_n)_-^{\alpha_0+2-2s}).$$

It will be the goal of this section to show this for general solutions  $u$ . The method will first involve a slight improvement of the barrier construction above to show  $u \in C^{\alpha_0}(E_1)$ , and then an argument in the spirit of N. Krylov's proof of the boundary Harnack principle [23] to show that the asymptotic form above actually holds. We begin with:

**Lemma 7.8.** *Let  $u$  be as in Theorem 7.7. Then if  $\alpha_0 \leq 1$ ,  $u \in C^{0,\alpha_0}(E_1)$ , while if  $\alpha_0 > 1$ ,  $u \in C^{1,\alpha_0-1}(E_1)$ . In either case,  $u \in C^{1,\alpha_0+1-2s}(E_1 \cap \Omega_1)$ .*

*Proof.* First, from the conclusion of Theorem 7.7, we have that  $u \in C^\gamma(E_{3/2} \cap \Omega_1)$  for some  $\gamma$  with  $\min\{\alpha_0 + 2 - 2s, 2s\} > \gamma > \alpha' > \alpha_0$ . Assume (without loss of generality from linearity) that  $|f| \leq 1$  and  $|u(x)| \leq |x_n|^\gamma$  on  $\Omega_1$ . We consider the following barrier function, claiming that it is a supersolution to (7.2):

$$\psi(x) = (1 + R^{\alpha'} + C_1)\rho_{\alpha_0}(x_n) + C_1 [-(\rho_{\alpha'}(x_n) \wedge R^{\alpha'}) + C_2\rho_\gamma(-x_n)] + \xi(|x'|),$$

where  $\xi$  is as in Lemma 7.5 (with  $r = 5/4$ , say) and  $C_1, C_2$  nonnegative with  $C_1 \cdot C_2 \geq 1$ . Then arguing as in that Lemma, it's clear that  $u \leq \psi$  on  $(E_{5/4} \cap \Omega_2)^c$ , and so the comparison argument given there applies to show  $u \leq \psi \leq (1 + R^{\alpha'} + C_1)|x_n|^{\alpha_0}$  in  $E_1 \cap \Omega_2$ . Now do the same for  $-u$ , combine with interior estimates, and apply Lemma 7.4 to conclude.

To see that  $\psi$  is indeed a supersolution, we first notice that by definition of  $\alpha_0$ , the first term is a solution. Indeed,

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{\rho_{\alpha_0}(x_n) - \rho_{\alpha_0}(y_n)}{|x - y|^{n+2s}} dy - (1 - \frac{\nu}{2}) \int_{\Omega_1} \frac{\rho_{\alpha_0}(x_n) - \rho_{\alpha_0}(y_n)}{|x - y|^{n+2s}} dy \\ &= A(n, s) \left[ q(s, \alpha_0) - (1 - \frac{\nu}{2}) \frac{1}{2s} \right] |x_n|^{\alpha_0-2s} = 0. \end{aligned}$$

The contribution from the other terms is then given by

$$\begin{aligned}
& \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{|x - y|^{n+2s}} dy - \left(1 - \frac{\nu}{2}\right) \int_{\Omega_1} \frac{\psi(x) - \psi(y)}{|x - y|^{n+2s}} dy \\
& \geq -C - C_1 o_R(1) \\
& \quad + C_1 \left[ -C_2 A(n, s) |x_n|^{\gamma-2s} - A(n, s) |x_n|^{\alpha'-2s} (q(s, \alpha') - q(s, \alpha_0)) \right] \\
& \geq 0,
\end{aligned}$$

with the last step from noting that as  $\alpha_0 < \alpha' < \gamma$ ,  $q(s, \alpha') > q(s, \alpha_0)$ , and then choosing  $C_2$  small,  $R$  large, and  $C_1$  large in that order.  $\square$

**Lemma 7.9.** *For each  $\sigma > 0$  there is a function  $B : \mathbb{R}^n \rightarrow \mathbb{R}$ , continuous and smooth away from  $\Gamma$ , with the following properties (here  $\gamma$  is as in Lemma 7.8):*

- $B(x) \leq -\rho_\gamma(-x_n)$  for  $x_n < 0$ ,  $x_n > 1 + \sigma$ , or  $|x'| \geq \sigma$ .
- $B(x) \leq C_* |x_n|^{\alpha_0}$  for  $x_n > 0$ .
- $B(x) \geq |x_n|^{\alpha_0}$  for  $1 > x_n > 0$  and  $|x'| \leq \frac{\sigma}{2}$ .
- $\int_{\mathbb{R}^n} \frac{B(x) - B(y)}{|x - y|^{n+2s}} dy - \left(1 - \frac{\nu}{2}\right) \int_{\Omega_1} \frac{B(x) - B(y)}{|x - y|^{n+2s}} dy \leq -1$  for  $x \in \{|x'| \leq \sigma, 0 < x_n < 1\}$ .

The constant depends on  $\sigma$ .

*Proof.* Set  $\xi : [0, \infty) \rightarrow [-1, 0]$  to be a smooth cutoff which vanishes on  $[0, \frac{\sigma}{2})$  and equals  $-1$  on  $[\sigma, \infty)$ . Set  $\varphi$  to be a smooth nonnegative function with mean 1 and supported on  $\{|x'| \leq \sigma, 1 + \sigma/2 \leq x_n \leq 1 + \sigma\}$ . Note that it is possible to have  $\|\xi\|_{C^2} \leq C\sigma^{-2}$  and  $\|\varphi\|_{L^\infty} \leq C\sigma^{-n}$ . Let

$$\begin{aligned}
B(x) = & \rho_{\alpha_0}(x_n) \wedge 2 + (2 + RC_2) [\xi(|x'|) + \xi((x_n - 1)_+)] + C_1 \varphi \\
& - \underbrace{\left[ \rho_\gamma(-x_n) - C_2 (\rho_{\alpha'}(x_n) \wedge R) \right]}_{(*)}
\end{aligned}$$

where the positive parameters  $C_1, C_2, R$  will be chosen below. Then the first, second, and third properties are clear and we just need to check  $B$  is a subsolution. The first term is, up to a bounded error, a solution

$$\begin{aligned}
& \int_{\Omega_2} \frac{\rho_{\alpha_0}(x_n) \wedge 2 - \rho_{\alpha_0}(y_n) \wedge 2}{|x - y|^{n+2s}} dy + \frac{\nu}{2} \int_{\Omega_1} \frac{\rho_{\alpha_0}(x_n) \wedge 2}{|x - y|^{n+2s}} dy \\
& \leq A(n, s) |x_n|^{\alpha_0-2s} \left[ q(s, \alpha_0) - \frac{1}{2s} \left(1 - \frac{\nu}{2}\right) \right] + C \\
& \leq C
\end{aligned}$$

provided  $x_n \leq 1$ . The function  $\xi(|x'|)$  is smooth and constant in the  $e_n$  direction, so as previously seen it contributes a term controlled by  $\sigma^{-2}$ , as does the other term  $\xi((x_n -$



1)<sub>+</sub>). The entire term (\*) was shown above to be a supersolution for  $R, C_2$  large enough. Then we are reduced to

$$\begin{aligned} \int_{\Omega_2} \frac{B(x) - B(y)}{|x - y|^{n+2s}} dy + \frac{\nu}{2} \int_{\Omega_1} \frac{B(x) - B(y)}{|x - y|^{n+2s}} dy &\leq C\sigma^{-2} + C_1 \int_{\Omega_2} \frac{-\varphi(y)}{|x - y|^{n+2s}} dy \\ &\leq C[\sigma^{-2} - C_1] \leq -1. \end{aligned}$$

provided  $C_1 \gg \sigma^{-2}$ . Set  $C_* \approx \sigma^{-2-n}$  to conclude.  $\square$

Note that the dependence on  $\sigma$  in the above lemma (the constant is  $\approx \sigma^{-2-n}$ ) is clearly not optimal, but this will not be relevant to the application below.

**Lemma 7.10.** *Let  $u$  satisfy (H). The quantity  $\frac{u(x)}{|x_n|^{\alpha_0}}$  extends continuously from  $E_1 \cap \Omega_2$  to  $\Gamma \cap E_1$ ; i.e. for every  $x \in \Gamma \cap E_1$ ,*

$$L(x) = \lim_{y_k \rightarrow x, y_k \in \Omega_2} \frac{u(y_k)}{(y_k)_n^{\alpha_0}}$$

*exists and is continuous. Indeed, the following stronger statement is true: for each  $x \in \Gamma \cap E_1$  and  $C_0 > 0$  there are constants  $\beta, C_1$  (independent of  $x$  and  $u$ ) such that*

$$\text{osc}_{B_r(x) \cap \Omega_2} \frac{u(y)}{y_n^{\alpha_0}} \leq C_2 r^\beta.$$

*Proof.* First, after a localization, an application of Lemma 7.8, and an initial rescaling  $u'(x) = R^{-\alpha_0} u(Rx)$  (which decreases the right-hand side  $f$  and otherwise leaves the equation (7.2) unchanged) it suffices to consider the case of  $|u(x)| \leq \rho_{\alpha_0}(x_n) + \eta \rho_\gamma(-x_n)$  and  $|f| \leq \eta$ .

We will show the following by induction on  $k$ :

$$\text{osc}_{E_{r^{-k}}} \frac{u(y)}{y_n^{\alpha_0}} \leq 2(1 - \Theta_1)^k$$

for some  $r, \Theta_1 > 0$ . Note this holds for  $k = 0$  by the reduction above. Suppose this holds for each  $l \leq k$ . Then define

$$v(x) = (1 - \Theta_1)^{-k} r^{-(k+1/2)\alpha_0} \left[ u(r^{-k-1/2}x) - m_k \rho_{\alpha_0}(x_n) \right],$$

where

$$m_k = \min_{E_{r^{-k}}} \frac{u}{y_n^{\alpha_0}}$$

From inductive assumption we have that  $0 \leq v/y_n^{\alpha_0} \leq 2$  on  $E_{r^{-1/2}} \cap \Omega_2$  and  $|v - y_n^{\alpha_0}| \leq y_n^{\alpha_0} + C(|x| - \frac{1}{2}r^{-1/2})_+^{\beta_0}$  for some  $\beta_0$  depending on  $r, \Theta_1$ . Assume without loss of generality that  $v(0, 1) \geq 1$  (otherwise consider  $2|x_n|^{\alpha_0} - v$ , which satisfies the same assumptions). Then applying Lemma 7.8, say, we have that  $v \geq \frac{1}{2}$  on  $B_{r_0}(0, 1)$ , where  $r_0$  is universal. By choosing  $r, \Theta_1$  small enough and fixing  $\zeta$  a smooth cutoff supported on  $E_{1/2r}$  and identically 1 on  $E_{1/4r}$ , it is easy to see that  $\zeta v$  satisfies

$$\zeta v(x) \geq -\eta \rho_\gamma(-x_n)$$

for all  $x$ , and moreover

$$\left| \int_{\Omega_2} \frac{\zeta v(x) - \zeta v(y)}{|x - y|^{n+2s}} dy + \frac{\nu}{2} \int_{\Omega_1} \frac{\zeta v(x) - \zeta v(y)}{|x - y|^{n+2s}} dy \right| \leq \eta$$

for  $x \in E_1$ . Let  $B$  be the function from Lemma 7.9 (with  $\sigma = r_0$ ) and  $C_* \geq 1$  the associated constant. Then for  $\eta$  small enough  $\tilde{B} = B/2C_* \leq \zeta v$  outside of  $\{|x'| \leq r_0, x_n \in [0, 1]\}$ , and is a strict subsolution with

$$\int_{\Omega_2} \frac{[\tilde{B}(x) - \zeta v(x)] - [\tilde{B}(y) - \zeta v(y)]}{|x - y|^{n+2s}} + \frac{\nu}{2} \int_{\Omega_1} \frac{[\tilde{B}(x) - \zeta v(x)] - [\tilde{B}(y) - \zeta v(y)]}{|x - y|^{n+2s}} \leq 0.$$

By the comparison argument in Lemma 7.5, it follows that  $\zeta v \geq \tilde{B}$  on all of  $\mathbb{R}^n$ , and so in particular  $\nu \geq \frac{1}{2C_*} x_n^\alpha$  on  $\{|x'| \leq \frac{r_0}{2}, x_n \in [0, 1]\}$ . Making sure  $r, \Theta_1$  are chosen so that  $r^{1/2} < \frac{r_0}{2}, \Theta_1 < \frac{1}{4C_*}$ , it follows that

$$\text{osc}_{E_{r^{1/2}}} \frac{v(y)}{y_n^{\alpha_0}} \leq 2(1 - \Theta_1),$$

which scales to

$$\text{osc}_{E_{r^{k+1}}} \frac{u(y)}{y_n^{\alpha_0}} \leq 2(1 - \Theta_1)^{k+1}.$$

The conclusion now follows from applying this to tangential translates of  $u$ .  $\square$

**Theorem 7.11.** *Let  $u$  satisfy (H). and  $|f|, |u| \leq 1$ . Then there is a number  $l \in \mathbb{R}$  such that*

$$u(x) = l[\rho_{\alpha_0}(x_n) + M_0 \rho_{\alpha_0+2-2s}(-x_n)] [1 + q(x)], \quad (7.4)$$

where  $|q(x)| \leq C|x|^\beta$  for some  $C, \beta$  independent of  $u$ .

Recall  $M_0$  was a negative constant depending only on  $s, \nu$  defined earlier:

$$M_0 = -\frac{\nu}{(2 + \alpha_0 - 2s)(1 + \alpha_0 - 2s)} A(n, s) \int_0^\infty \frac{t^{\alpha_0}}{(1 + t)^{1+2s}} dt$$

*Proof.* For  $x \in \Omega_2$ , this is an immediate consequence of Lemma 7.10, provided

$$l = \lim_{y \rightarrow 0, y \in \Omega_2} \frac{u(y)}{y_n^{\alpha_0}}.$$

We show that this implies improved regularity  $\Omega_1$ . Indeed, let

$$v(x) = u(x) - l[\rho_{\alpha_0}(x_n) + M_0 \rho_{\alpha_0+2-2s}(-x_n)].$$

Then from Lemma 7.10 and tangential regularity we have that for  $x \in \Omega_2$ ,

$$|v(x)| \leq C|x|^{\alpha_0+\beta}.$$

Now we check the equation on  $\Omega_1$  satisfied by  $v$ :

$$\begin{aligned} -\Delta v - f(x) + v \int_{\Omega_2} \frac{v(x) - v(y)}{|x - y|^{n+2s}} dy \\ = -l|x_n|^{\alpha_0-2s} \left[ -M_0(\alpha_0 + 2 - 2s)(\alpha_0 + 1 - 2s) - vA(n, s) \int_0^\infty \frac{t^{\alpha_0}}{(1+t)^{1+2s}} dt \right] \\ - \frac{lvM_0A(n, s)}{2s} |x_n|^{\alpha_0+2-4s}. \end{aligned}$$

The first term on the right vanishes from the definition of  $M_0$ , so we are left with

$$\begin{aligned} -\Delta v(x) &= f(x) + v \int_{\Omega_2} \frac{v(x) - v(y)}{|x - y|^{n+2s}} + O(|x_n|^{\alpha_0+2-4s}) \\ &= O(1 + |x_n|^{\alpha_0+2-4s}) + v \left[ \int_{\Omega_2} \frac{v(x) - v(0)}{|x - y|^{n+2s}} + \int_{\Omega_2} \frac{v(0) - v(y)}{|x - y|^{n+2s}} \right] \\ &= O\left(1 + |x|^{\alpha_0+\beta}|x_n|^{-2s} + |x_n|^{\alpha_0+2-4s}\right). \end{aligned}$$

Now the scaling argument in Lemma 7.4, performed only near 0, gives that  $|\nabla u(x)| \leq C|x|^\gamma$ , where  $\gamma = \min\{\alpha_0 + \beta + 1 - 2s, \alpha_0 + 3 - 4s\} > \alpha_0 + 1 - 2s$ . Integrating this easily gives

$$|v(x)| \leq C|x|^{1+\gamma}$$

for  $x \in \Omega_1$ , from which the conclusion follows immediately.  $\square$

*Remark 7.12.* The argument here can be continued to give a full asymptotic expansion for  $u$  up to terms with homogeneity greater than  $2s$ , where the profile is a multiple of  $\Phi$ , the solution constructed above. The relation (7.4) has a natural interpretation as a *transmission condition* for  $(P)$ : indeed, it implies that at every point on  $\Gamma \cap E_1$  we have

$$\lim_{t \rightarrow 0^+} \frac{u(x', t) - u(x', 0)}{t^{\alpha_0}} = \frac{1}{M_0} \lim_{t \rightarrow 0^+} \frac{u(x', -t) - u(x', 0)}{t^{\alpha_0+2-2s}}.$$

This is analogous to the classical transmission problem, in which the ratio of the (co)normal derivatives remains constant along the interface.

*Remark 7.13.* A straightforward modification of the techniques used here can be applied to the more general equation

$$\begin{aligned} \int f\psi &= \int_{\Omega_1} \langle \nabla \psi, \nabla u \rangle + \int_{\Omega_2} \int_{\Omega_2} \frac{[u(x) - u(y)][\psi(x) - \psi(y)] dx dy}{|x - y|^{n+2s}} \\ &\quad + \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)][v'\psi(x) - v\psi(y)]}{|x - y|^{n+2s}} dx dy \end{aligned}$$

(this reduces to the previous case when  $v = v'$ , and otherwise changes the equation over  $\Omega_1$ ), provided solutions exist and admit a Hölder estimate (which we do not show). Then the value  $\alpha_0$  remains exactly the same, as does the transmission condition, except the value of  $M_0$ , which is now given by

$$M_0 = -\frac{v'}{(2 + \alpha_0 - 2s)(1 + \alpha_0 - 2s)} A(n, s) \int_0^\infty \frac{t^{\alpha_0}}{(1+t)^{1+2s}} dt.$$

The rest of the asymptotic expansion changes analogously; observe in particular that the exponents do not depend on the value of  $\nu'$ . If  $\nu' = 0$ , the behavior is slightly different, in that now the equation over  $\Omega_1$  is just  $-\Delta u = f$ . Using the same argument as when  $\nu' > 0$ , we deduce that the normal derivative of  $u$  vanishes from  $\Omega_1$ . This means that  $u$  solves the Neumann problem for Laplace equation over  $\Omega_1$ , completely independently of what happens on  $\Omega_2$ . Thus it is simple to deduce the asymptotic form

$$u(x) = l\rho_{a_0}(x_n) + O(|x_n|^2 1_{\Omega_1} + |x_n|^{2s} 1_{\Omega_2})$$

near  $0 \in \Gamma$ . This calculation suggests that the  $\nu'$  term is of lower-order, and should not seriously effect the regularity theory. Unfortunately, it does break the variational structure of the equation, so some additional arguments would be required to prove the analogue of Theorem 5.5 in this case.

## 8 Near-Optimal Estimates for Flat Interface and Constant Coefficients

If the interface  $\Gamma$  is not flat, the argument given above will not suffice to prove higher regularity even in the simple case of  $A = I$ ,  $a = 1$ . The barriers used, and the way they were used, depended heavily on the special geometry of  $\Gamma$ . On the other hand, the localization and flattening of Lemma 5.8 can be applied to reduce to a situation where similar barriers are applicable. Over the course of the remaining sections, we show how to prove near-optimal regularity in the setting of general (smooth) coefficients and flat boundary.

Even in the simpler setting of a classical local transmission problem, the optimal (Lipschitz) regularity is to be expected only in the case that the coefficients and interface satisfy some minimal smoothness condition (say  $A \in C^{0,\alpha}$ ,  $\Gamma \in C^{1,\alpha}$ ). Indeed, the behavior of a transmission problem near the interface describes how solutions to a uniformly elliptic equation look like near jump-type discontinuities in the coefficient matrix. However, this is a meaningful situation to consider only if the coefficients are regular *elsewhere in the domain*. We will not strive to find optimal conditions on the coefficients for these regularity statements to hold, but the theory below will be sufficiently robust to deal with the example of a quasigeostrophic-type drift, as well as any problem with smooth parameters  $A, a, f, \Gamma$  with  $a$  symmetric. An effort will be made to discuss which of the assumptions made are *invariant under diffeomorphism*, and so preserved under the flattening procedure. When moving to the nonlocal case, there exists a second and less obvious obstruction. The matrix  $A$  can have, roughly speaking, only one type of regularity property: smoothness in the spacial dependence parameter  $x$ . Other structural constraints may be imposed, but in the extreme case of a translation-invariant matrix  $A$ , there ends up being no substantive difference with the situation of the Laplace operator treated above. For the nonlocal energy weight  $a(x, y)$ , the corresponding property of regularity under translation becomes smoothness of the mapping  $x \rightarrow a(z + x, z' + x)$ , with the extremal version being invariance under this operation. However, notice that the class of coefficients  $a$  that are invariant under translation is much richer in the nonlocal setting; indeed, there is another action

$x \rightarrow a(z + x, z' - x)$  which is separate from translation dependence. This captures the variation of  $a$  with respect to nonlocal “jump length,” when the equation is interpreted as a nonlocal diffusion.

There is no obvious reason regularity under this mapping needs to be assumed to ensure the regularity of solutions. However, without such assumptions it often becomes unavoidable to require control over the regularity of the boundary data in every estimate. Recently, some progress has been made on interior regularity for nonlocal equations without assumptions on the kernels’ dependence on this nonlocal parameter. See [23], or [26] for a simpler argument. In our case, however, some regularity in the nonlocal parameter seems required for the barrier construction above to function. Analogous considerations for boundary regularity for the nonlocal Dirichlet problem have been suggested by [27], and indeed it is likely that expected asymptotics near the boundary will hold only under some assumption of this type. A second reason to make this kind of restriction on  $a$  is that without it, structural properties are not preserved under diffeomorphism; this will be explained in detail in the following section.

With this discussion in mind, we outline the strategy we will pursue below. The argument is perturbative, along the lines of classical variational Schauder theory. The remainder of this section will be devoted to proving near-optimal estimates on solutions of the constant-coefficient equation. The next section will combine these estimates with a crude  $L^\infty$  stability statement and an iteration procedure. It will once again become important that the equation in question is not scale-invariant, and so all of the estimates need to be done independent of the scaling parameter  $\epsilon$ . In this section this will be rather trivial, as the effect of  $\epsilon$  is mainly on the transmission relation, not the optimal smoothness.

A final note before commencing: the case  $s = \frac{1}{2}$ ,  $\mathbf{b} \neq 0$  is perhaps the only one where the asymptotic behavior differs substantially from what was discussed above. Here  $\langle \mathbf{b}, e_n \rangle$  has the power to affect  $\alpha_0$  either favorably or unfavorably, depending on its sign.

We begin with the following situation:  $u \in C^{0,\alpha}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) \cap H^1(\Omega_1)$  is an admissible solution to  $(P_\epsilon)$  on  $E_2$ , with  $\Gamma = \{x_n = 0\}$ ,  $A(x) \equiv A$  constant, and  $a(x, y) = a(x, y)1_{\Omega_2 \times \Omega_2} + a(x, y)1_{\Omega_2 \times \Omega_1} + a(x, y)1_{\Omega_1 \times \Omega_2} := a^1(x, y) + a^2(x, y) + a^2(x, y)$  satisfying  $a^1(x + z, y + z) = a^1(x, y)$  for every  $x, y \in \Omega_2$  with  $x + z, y + z \in \Omega_2$ , and similarly  $a^2(x + z, y + z) = a^2(x, y)$  for all  $x, x + z \in \Omega_2$  and  $y, y + z \in \Omega_1$ . For simplicity we’ll use the notation  $a^i(x, y) = a_{s,i}(x - y)$ . We assume that  $a_{s,i}$  are uniformly elliptic; this implies in particular that  $a_{s,1}, a_{s,2}(z) \geq \lambda$ . Moreover, notice that  $a_{s,1}(z)$  is symmetric, meaning  $a_{s,1}(z) = a_{s,1}(-z)$ .  $\mathbf{b}(x) \equiv \mathbf{b}$  will remain fixed (and we will always consider only the cases of  $s \geq \frac{1}{2}$  or  $\mathbf{b} = 0$ ), and  $f(x)$  is bounded and smooth in the tangential directions.

It can be checked using the same method as for the fractional Laplace case that  $u$

satisfies the following equations:

$$\begin{cases} -\text{Tr}AD^2u(x) + \epsilon^{2(1-s)} \int_{\Omega_2} \frac{[u(x) - u(y)]a_{s,2}(x-y)}{|x-y|^{n+2s}} dy \\ \quad + \epsilon \langle \mathbf{b}, \nabla u(x) \rangle = \epsilon^2 f(x) & x \in E_2 \cap \Omega_1 \\ 2 \int_{\mathbb{R}^n} \frac{[u(x) - u(y)]a_{s,1}(x-y)}{|x-y|^{n+2s}} dy \\ \quad - \int_{\Omega_1} \frac{[u(x) - u(y)](2a_{s,1}(x-y) - a_{s,2}(x-y))}{|x-y|^{n+2s}} dy \\ \quad + \epsilon^{2s-1} \langle \mathbf{b}, \nabla u(x) \rangle = \epsilon^{2s} f(x) & x \in E_2 \cap \Omega_2 \end{cases}.$$

We justify that solutions have the expected tangential regularity up to the boundary. For this, we note that the following assumption on regularity of  $a_{s,i}$  is preserved under diffeomorphism, at least when the coefficients of the image are frozen at a point (this will be justified rigorously in the next section):

**Definition 8.1.** We say that  $a_{s,i}$  is in the class  $\mathcal{L}_k$  if there is a constant  $C$  such that  $a_{s,i}$  is  $k$  times continuously differentiable on its domain and

$$|D^\beta a_{s,i}(y)| \leq \frac{C}{|y|^{|\beta|}}$$

for each multi-index  $\beta$  with  $|\beta| \leq k$ . The class  $\mathcal{L}_1^*$  contains kernels satisfying the following stronger regularity criterion in the radial directions: there exists a modulus  $\omega(r)$  and a function  $a_{s,i}^{(0)} : S^{n-1} \rightarrow [0, \Lambda]$  such that for each  $r > 0$ ,  $\hat{y} \in S^{n-1}$ ,

$$|a_{s,i}(r\hat{y}) - a_{s,i}^{(0)}(\hat{y})| \leq \omega(r).$$

In addition,  $\omega$  satisfies the following condition (depending on  $s$ ):

$$\begin{cases} \lim_{r \rightarrow 0^+} \omega(r) = 0 & s < \frac{1}{2} \\ \int_0^1 \frac{\omega(r)}{r^{2s}} dr < \infty & s \geq \frac{1}{2} \end{cases}$$

A standard abuse of notation will be to extend  $a_{s,i}^{(0)}$  homogeneously (of degree 0) to  $\mathbb{R}^n$ . We have that  $a_{s,i}^{(0)} \geq \lambda$  for  $i = 1, 2$ .

**Lemma 8.2.** *If  $u$  is as above,  $e_j \perp e_n$  are tangential directions, and  $a_{s,i}$  are in  $\mathcal{L}_k$ , then*

$$\sup_{s>0} s^k \|u\|_{C^k(E_{3/2} \setminus \{|x_n| \leq s\})} + \|\partial_{e_1} \partial_{e_2} \cdots \partial_{e_k} u\|_{C^{0,\alpha}(E_{3/2})} \leq C(n, k, a) \|u\|_{L^\infty(\mathbb{R}^n)}.$$

*Proof.* (sketch) Let  $\eta \in C_c^\infty(E_t)$  a cutoff identically 1 on  $E_r$ , with  $3/2 < r < t < 2$ . The tangential regularity statement in the lemma will follow by induction from the following claim:

**Claim.** *For each direction  $e \perp e_n$ , and provided  $a_{s,i}$  are in  $\mathcal{L}_k$ , we have that  $\partial_e \eta u \in H^s(\mathbb{R}^n) \cap H^1(\Omega_1)$ . Furthermore,  $\partial_e u$  solves  $(P_\epsilon)$  on  $E_{r_1}$  with a right-hand side  $f$  which is  $C^{k-1}$  in the tangential directions, and  $\partial_e u$  is in  $C^{0,\alpha}(E_{r_1})$  for each  $r_1 < r$ .*

*Proof.* Let  $\delta_{e,h}w(x) = \frac{w(x+he)-w(x)}{h^\beta}$ . For any  $\phi \in C_c^\infty(E_{r_2})$ , use  $\delta_{e,-h}\phi$  as a test function for  $(P_\epsilon)$  for  $|h| \leq \frac{r-r_2}{4}$  and  $r_1 < r_2 < r$ . This gives

$$\begin{aligned} 0 &= B_L[u, \delta_{e,-h}\phi] + \epsilon^{2(1-s)} B_N[u, \delta_{e,-h}\phi] \\ &\quad - \epsilon \int u \langle \mathbf{b}, \nabla \delta_{e,-h}\phi \rangle - \epsilon^2 \int f \delta_{e,-h}\phi \\ &= B_L[\delta_{e,h}\eta u, \phi] + \epsilon^{2(1-s)} B_N[\delta_{e,h}\eta u, \phi] \\ &\quad - \epsilon \int \delta_{e,h}\eta u \langle \mathbf{b}, \nabla \phi \rangle - \epsilon^2 \int \delta_{e,h}f\phi + \epsilon^{2(1-s)} B_N[\delta_{e,h}(1-\eta)u, \phi] \end{aligned}$$

The rightmost term can be re-expressed as

$$\begin{aligned} &= \int_{\Omega_2 \times \Omega_2} \frac{a_{s,1}(x-y) [\delta_{e,h}(1-\eta)u(x) - \delta_{e,h}(1-\eta)u(y)] [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dx dy \\ &\quad + \int_{\Omega_2 \times \Omega_1} a_{s,i}(x-y) \cdots dx dy \\ &= \int_{\Omega_2 \times \Omega_2} \frac{-2a_{s,1}(x-y)\delta_{e,h}[(1-\eta)u](x)dx\phi(y)dy}{|x-y|^{n+2s}} + \int_{\Omega_2 \times \Omega_1} a_{s,i}(x-y) \cdots dx dy \\ &= -2 \int_{\Omega_2 \times \Omega_2} \delta_{e,h} \left( \frac{a_{s,1}}{|x-y|^{n+2s}} \right) (1-\eta)u(x)dx\phi(y)dy - \cdots \\ &= \int f_h(x)\phi(x)dx, \end{aligned}$$

with  $f_h$  bounded uniformly in  $h$  and  $C^{k-1}$  in the tangential directions (the ellipsis represents the other two completely analogous terms). We used the fact that the supports of  $\phi$  and  $(1-\eta)$  are bounded away from each other, and that the kernels  $\frac{a_{s,i}(x-y)}{|x-y|^{n+2s}}$  have integrable derivatives for  $|x-y|$  bounded away from 0. The third step was an integration by parts, enabled by the fact that  $e \perp e_n$ . A similar computation for the approximated problems  $(P_\epsilon^\delta)$  gives that  $\delta_{e,h}u$  is an admissible solution to  $(P_\epsilon)$  with this right-hand side.

Also we have that  $\delta_{e,h}\eta u \in L^\infty(\mathbb{R}^n)$  provided  $\beta = \alpha$  (the Hölder modulus for  $u$  from Theorem 5.5), so it follows from the energy estimates and Theorem 5.5 that

$$\|\delta_h \eta u\|_{C^{0,\alpha}(E_{r_1}) \cap H^1(\Omega_1) \cap H^s(\mathbb{R}^n)} \leq C(\mathbf{b}, f, \|u\|_{C^{0,\alpha}}, \lambda, \Lambda, r_1, r_2, r) = C.$$

From standard facts about difference quotients (see [7, Lemma 5.6]), this implies that  $|u(x+eh) - u(x)| \leq C|h|^{2\alpha}$ , and so the same procedure can be performed for  $\beta = 2\alpha$ , etc. until  $\beta = 1$ , at which point the claim follows. Note that the  $\alpha$  gained in each iteration is the same, so the procedure terminates after finitely many steps.  $\square$

Apply the claim inductively, choosing a sequence of nested cylinders, say  $E_{3/2+2^{-k}}$ . For the weighted interior estimate, simply observe that the argument given here applies for *any* direction provided that we restrict to  $x \in \{x_n < -1/2\}$  or  $x \in \{x_n > 1/2\}$ ; the  $s$  dependence follows from scaling.  $\square$

Next we would like make the same reduction to situation  $(H)$  as in the fractional Laplace case, and study the regularity of  $u$  in the  $e_n$  direction. There is, however, a

geometric obstruction. In the case of the local transmission problem, this takes the following form: the transmission condition is best expressed as a jump in the *conormal derivative* of  $u$ . However, the coefficients on the left and right may have different conormal directions, so while it is always possible to re-express this as a jump in the normal derivative, this expression will depend on the local tangential behavior of  $u$ . Nevertheless, it is easy to see (if the coefficients are constant) that after applying a specific piecewise-linear transformation to the domain,  $u$  solves Laplace equation on each side, and so characterizing regularity becomes trivial.

In our setting, it will not always be possible to recover the fractional Laplace case treated previously by a coordinate change. However, it is possible to reduce the problem to a more isotropic setting on each side separately, which is enough for the barrier argument to apply. On  $\Omega_1$  bootstrap regularity is generally simpler, and the reduction to  $(H)$  is not truly needed.

The following spherical averages of the homogeneous parts of  $a_{s,i}$  will appear frequently; as we will see they characterize the optimal regularity of the problem.

$$A_{s,1} = \int_{\mathbb{R}^{n-1}} \frac{a_{s,1}^{(0)}(y', 1)}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \geq A(n, s)\lambda > 0$$

$$A_{s,2} = \int_{\mathbb{R}^{n-1}} \frac{a_{s,2}^{(0)}(y', 1)}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \geq A(n, s)\lambda > 0$$

Also, define the following spherical moments (these are vectors in  $\mathbb{R}^{n-1}$ ):

$$M_{s,1} = \int_{\mathbb{R}^{n-1}} \frac{a_{s,1}^{(0)}(y', 1)y'}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy'$$

$$M_{s,2} = \int_{\mathbb{R}^{n-1}} \frac{a_{s,2}^{(0)}(y', 1)y'}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy'$$

**Definition 8.3.** The vector  $v_1 = \frac{(Ae_n)'}{\langle e_n, Ae_n \rangle} \in \mathbb{R}^{n-1}$  is called the *conormal ratio* of  $A$  (where  $(Ae_n)'$  is the projection of  $Ae_n$  onto the orthogonal complement of  $e_n$ ). If  $a_{s,i}$  are in  $\mathcal{L}_1^*$ , the vector  $v_2 \in \mathbb{R}^{n-1}$  is defined as

$$\begin{cases} v_2 = \frac{v_1 A_{s,1} \frac{1}{2s} + M_{s,2} - 2M_{s,1}}{2A_{s,1} - \frac{2s-1}{2s} A_{s,2}} & s > \frac{1}{2} \\ v_2 = 0 & s \leq \frac{1}{2} \end{cases} \quad (8.1)$$

unless the denominator in the first formula vanishes, in which case set  $v_2 = 0$  as well. This  $v_2$  is called the *effective nonlocal conormal ratio* of  $a$ .

This definition may seem arbitrary, but as the rest of this section will show  $v_2$  plays a role analogous to  $v_1$  for the nonlocal form  $B_N$ . The fact that  $v_2$  depends on  $v_1$  is a relic of the fact that the transmission term sees the local part of the equation; in general  $v_2$  measures to what extent and in what direction  $a$  fails to be isotropic near  $\Gamma$ . The next lemma will demonstrate that in the situations when  $v_2$  is relevant (namely, when the expected Hölder exponent  $\alpha_0 > 1$ , the denominator in the above expression will be strictly negative.



The next step is to perform a more subtle barrier construction which is appropriate in this anisotropic setting.

**Lemma 8.4.** Assume  $u$  solves  $(P_\epsilon)$ ,  $\mathbf{b}$ ,  $f$ ,  $A$ ,  $a$  are as above, and  $a_{s,i}$  are in  $\mathcal{L}_2 \cap \mathcal{L}_1^*$ . Define  $\alpha_0$  to be the unique solution in  $((2s-1)_+, 2s)$  to the equation

$$\begin{cases} 2 \left[ q(s, \alpha_0) - \frac{1}{2s} \right] A_{s,1} + \frac{1}{2s} A_{s,2} = 0 & s \neq \frac{1}{2} \\ 2 \left[ q(s, \alpha_0) - \frac{1}{2s} \right] A_{s,1} + \frac{1}{2s} A_{s,2} = -\alpha_0 \langle \mathbf{b}, e_n \rangle & s = \frac{1}{2} \end{cases}.$$

Then the following hold:

1. If  $u \in C^{0,\alpha}(E_r \cap \bar{\Omega}_1)$ ,  $\alpha_1 < \min\{\alpha, \alpha_0\}$ , then  $u \in C^{0,\alpha_1}(E_1 \cap \bar{\Omega}_2)$ .
2. If  $u \in C^{1,\alpha-1}(E_r \cap \bar{\Omega}_1)$ ,  $\alpha_0 > 1$ , and  $\partial_{Ae_n} u(x', 0^-) = 0$  (from  $\Omega_1$ ), then  $u \in C^{1,\alpha_1-1}(E_1 \cap \bar{\Omega}_2)$  for each  $1 < \alpha_1 < \min\{\alpha, \alpha_0\}$ , and  $\partial_{(v_2,1)} u(x', 0) = 0$ .

*Proof.* Localize as in Lemma 5.6 so the hypotheses are true globally. We construct a barrier of the form

$$w(x', x_n) = C_1 \rho_\alpha(-x_n) + C_2 \rho_{\alpha_1}(x_n) + \begin{cases} u(x' + v_1 x_n, 0) & x_n \leq 0 \\ u(x' - v_2 x_n, 0) & x_n \geq 0 \end{cases}.$$

Checking that this works is rather laborious; for convenience we rewrite the equation satisfied on  $\Omega_2$  as

$$\mathcal{L}u(x) + \epsilon^{2s-1} \langle \mathbf{b}, \nabla u(x) \rangle = \epsilon^{2s} f(x).$$

We will show that

$$\mathcal{L}w(x) + \epsilon^{2s-1} \langle \mathbf{b}, \nabla w(x) \rangle \geq C|x_n|^{\alpha_1-2s}$$

for  $|x_n|$  small enough; then by making sure  $C_1, C_2$  are large enough so that  $w \geq u$  on  $(E_1 \cap \Omega_2 \cap \{x_n < \delta\})^c$ , we can argue as in Lemma 7.5 to conclude that  $u \leq w$  on  $\mathbb{R}^n$ , and so in particular (after applying to  $\pm u$ )  $|u(x', x_n) - u(x - v_2 x_n, 0)| \leq C_2 |x_n|^{\alpha_1}$  for  $x_n > 0$ .

Now for the computation. First the  $\rho_\alpha$  term ( $x$  will always be in  $\Omega_2 \cap E_1$ ):

$$\mathcal{L}\rho_\alpha(x) = - \int_{\Omega_1} \frac{a_{s,2}(x-y)\rho_\alpha(-y_n)dy}{|x-y|^{n+2s}} \geq -C(n, s, \Lambda)|x_n|^{\alpha-2s}.$$

Now for  $\rho_{\alpha_1}$ :

$$\begin{aligned} \mathcal{L}\rho_{\alpha_1}(x) &= 2 \int_{\mathbb{R}^n} \frac{a_{s,1}(x-y)[\rho_{\alpha_1}(x) - \rho_{\alpha_1}(y)] dy}{|x-y|^{n+2s}} \\ &\quad - \int_{\Omega_1} \frac{\rho_{\alpha_1}(x)(2a_{s,1}(x-y) - a_{s,2}(x-y))}{|x-y|^{n+2s}} dy \\ &= |x_n|^{\alpha_1-2s} \left[ \int_{\Omega_2} \frac{a_{s,2}(x_n y', x_n(1+y_n))}{((1+y_n)^2 + |y'|^2)^{\frac{n+2s}{2}}} dy + 2 \int_{\Omega_2} \frac{(1-y_n^{\alpha_1})a_{s,1}(x_n y', x_n(1-y_n))}{((1-y_n)^2 + |y'|^2)^{\frac{n+2s}{2}}} dy \right] \\ &= |x_n|^{\alpha_1-2s} \left[ \int_0^\infty \frac{1}{(1+y_n)^{1+2s}} \int_{\mathbb{R}^{n-1}} \frac{a_{s,2}(x_n y'(1+y_n), x_n(1+y_n))}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' dy_n + \right. \\ &\quad \left. + 2 \int_0^\infty \frac{1-y_n^{\alpha_1}}{(1-y_n)^{1+2s}} \int_{\mathbb{R}^{n-1}} \frac{a_{s,1}(x_n y'(1-y_n), x_n(1-y_n))}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' dy_n \right]. \end{aligned}$$

At this point make use of the expansion for  $a_{s,1}$  as  $a_{s,1}(r\hat{y}) = a_{s,1}^{(0)}(\hat{y}) + \omega(r \wedge 1)$ , and similarly for  $a_{s,2}$  :

$$\begin{aligned}\mathcal{L}\rho_{\alpha_1}(x) &= |x_n|^{\alpha_1-2s} \left[ \int_0^\infty \frac{1}{(1+y_n)^{1+2s}} \int_{\mathbb{R}^{n-1}} \frac{a_{s,2}^{(0)}(y', 1)}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' dy_n \right. \\ &\quad \left. + 2 \int_0^\infty \frac{1-y_n^{\alpha_1}}{(1-y_n)^{1+2s}} \int_{\mathbb{R}^{n-1}} \frac{a_{s,1}^{(0)}(y', 1)}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' dy_n \right] + O(|x_n|^{\alpha_1-2s} \omega(|x_n|)) \\ &= \left\{ 2 \left[ q(s, \alpha_1) - \frac{1}{2s} \right] A_{s,1} + \frac{1}{2s} A_{s,2} \right\} |x_n|^{\alpha_1-2s} + O(|x_n|^{\alpha_1-2s} \omega(|x_n|)).\end{aligned}$$

The symmetry of  $a_{s,1}$  was used in this step to say that  $a_{s,1}^{(0)}(y', 1) = a_{s,1}^{(0)}(y', -1)$ . The drift term has a top-order contribution only when  $s = \frac{1}{2}$  (recall the standing assumption that if  $s < \frac{1}{2}$  then  $\mathbf{b} = 0$ ):

$$\langle \mathbf{b}, \nabla \rho_{\alpha_1}(x) \rangle = \alpha_1 \langle \mathbf{b}, e_n \rangle |x_n|^{\alpha_1-1}.$$

Finally we estimate the third term, denoted by  $v$ . This turns out to be somewhat more delicate if  $s > \frac{1}{2}$ ; if  $s < \frac{1}{2}$  then  $\mathcal{L}v$  is easily seen to be bounded from the fact that  $v$  is Lipschitz, while if  $s = \frac{1}{2}$ , the same gives that  $|\mathcal{L}v| \leq C|\log x_n|$ , which is still much smaller than  $|x_n|^{\alpha_1-2s}$ . Note that in these cases  $v_2$  could have been arbitrary. For  $s > \frac{1}{2}$ , we expand  $v$  as a Taylor series near  $x \in \Omega_2$  in the following way:

$$\begin{aligned}v(x) - v(y) &= \langle \nabla v(x', 0^+), x - y \rangle + O(|x - y|^2 + x_n^2) \\ &= \langle \nabla' u(x', 0), x' - y' + v_2(x_n - y_n) \rangle + O(|x - y|^2 + x_n^2)\end{aligned}$$

provided  $y \in \Omega_2$  (here  $\nabla'$  stands for the  $n-1$  dimensional gradient along  $\Gamma$ ). For  $y \in \Omega_1$ , we use the following instead:

$$\begin{aligned}v(x) - v(y) &= v(x) - v(x', 0) + v(x', 0) - v(y) \\ &= \langle \nabla v(x', 0^+), x - (x', 0) \rangle + \langle \nabla v(x', 0^-), (x', 0) - y \rangle + O(|x - y|^2 + x_n^2) \\ &= \langle \nabla' u(x', 0), v_2 x_n + x' - y' + v_1 y_n \rangle + O(|x - y|^2 + x_n^2).\end{aligned}$$

This enables us to reduce the expression for  $\mathcal{L}v$  to a simpler form:

$$\begin{aligned}\mathcal{L}v(x) &= 2 \int_{\Omega_2} \frac{[v(x) - v(y)]a_{s,1}(x - y)}{|x - y|^{n+2s}} dy + \int_{\Omega_1} \frac{[v(x) - v(y)]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy \\ &= 2 \int_{E_{3/2} \cap \{y_n > 2x_n\}} \frac{[v(x) - v(y)]a_{s,1}(x - y)}{|x - y|^{n+2s}} dy \\ &\quad + \int_{\Omega_1 \cap E_{3/2}} \frac{[v(x) - v(y)]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy + O(1) \\ &= 2 \int_{\{y_n > 2x_n\} \cap E_{3/2}} \frac{a_{s,1}(x - y) \langle \nabla' u(x', 0), x' - y' + v_2(x_n - y_n) \rangle}{|x - y|^{n+2s}} dy \\ &\quad + \int_{\Omega_1 \cap E_{3/2}} \frac{a_{s,2}(x - y) \langle \nabla' u(x', 0), x' - y' + v_2 x_n + v_1 y_n \rangle}{|x - y|^{n+2s}} dy + O(1).\end{aligned}$$

The first step used the fact that the contributions from outside of  $E_{3/2}$  are  $O(1)$ , while the symmetry of  $a_{s,1}$  ensures that the integral over  $\{0 < y_n < 2x_n\}$  vanishes, as the contributions from the strips  $\{0 < y_n < x_n\}$  and  $\{x_n < y_n < 2x_n\}$  cancel. The second step observes that as in both integrals,  $|x_n - y| \geq x_n$ , the integral of something of order  $O(|x - y|^2 + x_n^2)$  against the kernel gives a contribution of  $O(1)$ . We now proceed by changing variables in both terms:

$$\begin{aligned}
\mathcal{L}v(x) &= 2 \int_{\{y_n > x_n\} \cap E_{1/2}} \frac{a_{s,1}(y) \langle \nabla' u(x', 0), -y' - v_2 y_n \rangle}{|y|^{n+2s}} dy \\
&\quad + \int_{\{y_n > x_n\} \cap E_{1/2}} \frac{a_{s,2}(y) \langle \nabla' u(x', 0), y' + v_2 x_n + v_1(x_n - y_n) \rangle}{|y|^{n+2s}} dy + O(1) \\
&= 2 \int_{\{y_n > x_n\} \cap E_{1/2}} \frac{a_{s,1}^{(0)}(y) \langle \nabla' u(x', 0), -y' - v_2 y_n \rangle}{|y|^{n+2s}} dy \\
&\quad + \int_{\{y_n > x_n\} \cap E_{1/2}} \frac{a_{s,2}^{(0)}(y) \langle \nabla' u(x', 0), y' + v_2 x_n + v_1(x_n - y_n) \rangle}{|y|^{n+2s}} dy \\
&\quad + O(1 + \int_{\{y_n > x_n\} \cap E_{1/2}} \frac{\omega(|y|)(y' + y_n + x_n)}{|y|^{n+2s}} dy).
\end{aligned}$$

This last integral is easily seen to be of order  $O(1)$  from the assumptions on the modulus  $\omega$ . Now we may factor out the  $y_n$  dependence and change variables, to obtain

$$\begin{aligned}
\mathcal{L}v(x) &= \int_{x_n}^{1/2} \left[ y_n^{-2s} \int_{B_{1/(2y_n)}} \frac{2a_{s,1}^{(0)}(y', 1) \langle \nabla' u(x', 0), -y' - v_2 \rangle}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \right. \\
&\quad + y_n^{-2s} \int_{B_{1/(2y_n)}} \frac{a_{s,2}^{(0)}(y', 1) \langle \nabla' u(x', 0), y' - v_1 \rangle}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \\
&\quad \left. + y_n^{-2s-1} x_n \int_{B_{1/(2y_n)}} \frac{a_{s,2}^{(0)}(y', 1) \langle \nabla' u(x', 0), v_2 + v_1 \rangle}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \right] dy_n + O(1) \\
&= \int_{x_n}^{1/2} y_n^{-2s} \langle \nabla' u, I_1(1/2y_n) \rangle + y_n^{-2s-1} x_n I_2(1/2y_n) dy_n + O(1), \tag{8.2}
\end{aligned}$$

where

$$\begin{cases} I_1(R) = \int_{B_{1/R}} \frac{2a_{s,1}^{(0)}(y', 1)(-y' - v_2) + a_{s,2}^{(0)}(y', 1)(y' - v_1)}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \\ I_2(R) = \int_{B_{1/R}} \frac{a_{s,2}^{(0)}(y', 1)(v_2 + v_1)}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \end{cases}$$

are vector-valued. Notice that the integrals  $I_1(R)$ ,  $I_2(R)$  are convergent, bounded uniformly in  $R$ , and that  $I_1(R) \rightarrow -2M_{s,1} + M_{s,2} - 2A_{s,1}v_2 - A_{s,2}v_1 = I_1(\infty)$  as  $R \rightarrow \infty$ , while  $I_2(R) \rightarrow A_{s,2}(v_1 + v_2) = I_2(\infty)$ . In the case of  $\alpha_0 \leq 1$ , it is sufficient to observe that (only using the boundedness of  $I_j(R)$ ) we obtain  $\mathcal{L}v(x) = O(|x_n|^{1-2s})$ , and that  $1 - 2s \geq \alpha_0 - 2s > \alpha' - 2s$  and hence lower order; a much cruder argument would have

been enough in this regime. If  $\alpha_0 > 1$ , we first claim that  $I_j(R)$  converge very rapidly to their limits. Indeed, using the obvious estimates

$$\int_{|y'|>R} \frac{1}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' \leq CR^{-1-2s}$$

and

$$\int_{|y'|>R} \frac{|y'|}{(1+|y'|^2)^{\frac{n+2s}{2}}} dy' \leq CR^{-2s}$$

gives

$$|I_1(R) - I_1(\infty)| \leq CR^{-2s},$$

while

$$|I_2(R) - I_2(\infty)| \leq CR^{-2s-1}.$$

Substituting this into equation 8.2,

$$\begin{aligned} \mathcal{L}v &= O(1) + \int_{x_n}^{1/2} y_n^{-2s} I_1(\infty) + x_n y_n^{-2s-1} I_2(\infty) dy_n \\ &= O(1) + x_n^{1-2s} \left[ \frac{1}{2s-1} I_1(\infty) + \frac{1}{2s} I_2(\infty) \right]. \end{aligned}$$

Now we may compute

$$\begin{aligned} \frac{1}{2s-1} I_1(\infty) + \frac{1}{2s} I_2(\infty) &= \frac{-2M_{s,1} + M_{s,2} - 2A_{s,1}\nu_2 - A_{s,2}\nu_1}{2s-1} + \frac{A_{s,2}(\nu_1 + \nu_2)}{2s} \\ &= \nu_2 \left[ \frac{-2}{2s-1} A_{s,1} + \frac{1}{2s} A_{s,2} \right] + \nu_1 A_{s,2} \frac{1}{2s(2s-1)} + \frac{M_{s,2} - 2M_{s,1}}{2s-1} = 0 \end{aligned}$$

by the definition of  $\nu_2$ . Notice that as  $\alpha_0 > 1$ ,  $q(s, \alpha_0) > q(s, 1) = \frac{1}{1-2s} + \frac{1}{2s}$  gives that the denominator in the expression for  $\nu_2$  is strictly less than 0, and so, in particular,  $\nu_2$  is well-defined. Thus  $\mathcal{L}v = O(1)$ , as desired. Also,  $v$  has bounded derivatives, so the drift term  $\langle \mathbf{b}, \nabla v \rangle$  is bounded.

Putting everything together, we have that (with the term in braces only if  $s = \frac{1}{2}$ ),

$$\begin{aligned} \mathcal{L}w + \epsilon^{2s-1} \langle \mathbf{b}, \nabla w(x) \rangle &\geq \\ &\geq C_2 |x_n|^{\alpha_1-2s} \left[ 2 \left( q(s, \alpha_1) - \frac{1}{2s} \right) A_{s,1} + \frac{1}{2s} A_{s,2} + \{ \alpha_1 \langle \mathbf{b}, e_n \rangle \} \right] \\ &\quad - O(|x_n|^{\alpha_1-2s} \omega(x_n) + |x_n|^{\alpha-2s} + 1). \end{aligned}$$

Since  $\alpha_1 < \alpha_0$ , the term in brackets is positive, and so by choosing  $C_2$  large the entire right-hand side is positive for  $|x_n|$  small.

So far we have shown that  $|u(x, x_n) - v(x, x_n)| \leq C|x_n|^{\alpha_1}$  globally. We also have that  $v_1 = u - v$  admits the following trivial estimate:

$$\left| \int_{\mathbb{R}^n} \frac{a_{s,1}(x-y)[v_1(x) - v_1(y)]}{|x-y|^{n+2s}} dy \right| \leq C|x_n|^{-2s}.$$

This is true for  $v$  because it is smooth on  $\Omega_2$ , while for  $u$  it follows from using the equation. Therefore from an easy scaling argument the following interior estimates are available for  $x, y \in E_1 \cap \Omega_2$  and  $|x - y| < \frac{1}{2}x_n$ :

$$|v_1(x) - v_1(y)| \leq C|x - y|x_n^{\alpha_1 - 1}$$

and

$$|\nabla v_1(x) - \nabla v_1(y)| \leq C|x - y|^{2s-1}x_n^{\alpha_1 - 2s}.$$

If  $\alpha_1 \leq 1$ , use the first estimate when  $|x - y| \leq \frac{1}{2} \max\{x_n, y_n\}$  to get

$$|v_1(x) - v_1(y)| \leq C|x - y|^{\alpha_1},$$

or else use the estimate from the barrier directly to give

$$|v_1(x) - v_1(y)| \leq |v_1(x)| + |v_1(y)| \leq C[|x_n|^{\alpha_1} + |y_n|^{\alpha_1}] \leq C|x - y|^{\alpha_1}.$$

If  $\alpha_1 > 1$ , use the second estimate when it applies to get

$$|\nabla v_1(x) - \nabla v_1(y)| \leq C|x - y|^{\alpha_1 - 1}.$$

Then when  $|x - y| > \frac{1}{2} \max\{x_n, y_n\}$  we have

$$|\nabla v_1(x) - \nabla v_1(y)| \leq |\nabla v_1(x)| + |\nabla v_1(y)|,$$

and these can be estimated as

$$\begin{aligned} |\nabla v_1(x)| &= |\nabla v_1(x) - \nabla v_1(x', 0)| \\ &\leq \sum_{k=0}^{\infty} |\nabla v_1(x, 2^{-k}x_n) - \nabla v_1(x, 2^{-k-1}x_n)| \\ &\leq C \sum_{k=0}^{\infty} 2^{-k(\alpha_1 - 1)} |x_n|^{\alpha_1 - 1} \\ &\leq C|x - y|^{\alpha_1 - 1}. \end{aligned}$$

This shows that  $v_1$ , and hence  $u$ , is in  $C^{0, \alpha_1}(E_1 \cap \Omega_2)$  (or  $C^{1, \alpha_1 - 1}(E_1 \cap \Omega_2)$ ).  $\square$

Next is a corresponding argument for the local side. Here, as we will see, using the first-order vanishing of  $\partial_{(v_2, 1)}u$  on  $\Omega_2$  does not always work, since it requires a compatibility of the conormal vectors. This is made precise in the following definition:

**Definition 8.5.** The collection  $A, a_{s,1}, a_{s,2}$  is called *compatible* if the following relation holds:

$$\frac{1}{2s}v_2A_{s,2} - \frac{2s-1}{2s}v_1A_{s,2} + M_{s,2} = 0.$$

Compatibility can be thought of as the condition needed such that there is a global reduction to situation (H). In other words, that after composition with a piecewise linear transformation of  $\mathbb{R}^n$ ,  $u$  solves an equation with both  $v_1, v_2 = 0$  and in addition  $a_{s,2}$  is isotropic in an averaged sense.

**Lemma 8.6.** *Let  $u$  be a solution to  $(P_\epsilon)$  and  $r > 1$ . Then we have:*

1. *If  $u \in C^{0,\alpha}(E_r)$  then  $u \in C^{0,(\alpha+2-2s)\wedge 1}(E_1 \cap \bar{\Omega}_1)$ .*
2. *If  $u \in C^{0,\alpha}(E_r)$ ,  $2 > \alpha + 2 - 2s > 1$ , and in addition  $\partial_{Ae_n} u(x', 0^-) = 0$  (from  $\Omega_1$ , in the sense of distributions), then  $u \in C^{1,\alpha+1-2s}(E_1 \cap \bar{\Omega}_1)$ .*
3. *If  $u \in C^{1,\alpha}(E_r \cap \bar{\Omega}_2) \cap C^{1,\alpha}(E_r \cap \bar{\Omega}_1)$  and  $\partial_{Ae_n} u(x', 0^-) = 0$ ,  $\partial_{(v_2,1)} u(x', 0^+) = 0$ ,  $a_{s,2} \in \mathcal{L}_1^*$ , and in addition  $a_{s,i}, A$  are compatible, then  $u \in C^{1,\alpha'+2-2s}(E_1 \cap \bar{\Omega}_1)$  for all  $\alpha' \leq \alpha$  with  $\alpha' < 2s - 1$ .*

*Proof.* The proof is analogous to that of Lemma 7.4 (it is now necessary to subtract planes in the tangential direction before scaling, which doesn't effect the equation); we only show how to obtain estimates on the nonlocal term. For (1) and (2), the following basic estimate is sufficient:

$$\left| \int_{\Omega_2} \frac{[u(x) - u(y)]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy \right| \leq C \int_{\Omega_2} |x - y|^{\alpha-n-2s} dy \leq C|x_n|^{\alpha-2s}.$$

If in situation (3), the computation is more subtle; we expand  $u(y) = u(y' + v_2 y_n, 0) + O(|y_n|^{1+\alpha} \wedge 1)$ ,  $u(x) = u(x' - v_1 x_n, 0) + O(|x_n|^{1+\alpha} \wedge 1)$ :

$$\begin{aligned} & \int_{\Omega_2} \frac{[u(x) - u(y)]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy \\ &= \int_{\Omega_2 \cap E_{3/2}} \frac{[u(x' - v_1 x_n, 0) - u(y' + v_2 y_n, 0)]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy + O(|x_n|^{\alpha+1-2s}) \\ &= \int_{\Omega_2 \cap E_{3/2}} \frac{[\langle \nabla' u(x' - v_1 x_n, 0), x' - y' - v_2 y_n - v_1 x_n \rangle]a_{s,2}(x - y)}{|x - y|^{n+2s}} dy \\ &\quad + \int_{\Omega_2 \cap E_{3/2}} \frac{O(|y' - x' + v_2 y_n + v_1 x_n|^2)}{|x - y|^{n+2s}} dy + O(|x_n|^{\alpha+1-2s}) \end{aligned}$$

We claim that the second term is lower-order (using the fact that  $s > \frac{1}{2}$ ):

$$\begin{aligned} & \int_{\Omega_2 \cap E_{3/2}} \frac{|y' - x' + v_2 y_n + v_1 x_n|^2 a_{s,2}(x - y)}{|x - y|^{n+2s}} dy \\ & \leq C \int_0^{3/2} \frac{(x_n + y_n)^2}{(x_n + y_n)^{1+2s}} dy_n \int_{\mathbb{R}^{n-1}} \frac{(1 + |y'|)^2}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' \leq C. \end{aligned}$$

Back to the original computation,

$$\begin{aligned} &= \int_{\Omega_2 \cap E_{3/2}} \frac{\langle \nabla' u(x' - v_1 x_n, 0), x' - y' - v_2 y_n - v_1 x_n \rangle a_{s,2}(x - y)}{|x - y|^{n+2s}} dy + O(|x_n|^{\alpha+1-2s}) \\ &= \int_{\{y_n > x_n\} \cap B_{1/2}} \frac{\langle \nabla' u, y' - v_2(x_n - y_n) - v_1 x_n \rangle a_{s,2}(y)}{|y|^{n+2s}} dy + O(|x_n|^{\alpha+1-2s}) \\ &= \int_{x_n}^{1/2} |y_n|^{-2s} \int_{B_{1/2y_n}} \frac{\langle \nabla' u, y' - v_2(x_n/y_n - 1) - v_1 x_n/y_n \rangle a_{s,2}^{(0)}(y', -1)}{(1 + |y'|^2)^{\frac{n+2s}{2}}} dy' dy_n \\ &\quad + O(|x_n|^{\alpha+1-2s}). \end{aligned}$$

At this point, considerations as in the previous lemma give that this integral is  $O(1)$  provided

$$v_2 \frac{1}{2s} A_{s,2} - \frac{2s-1}{2s} v_1 A_{s,2} + M_{s,2} = 0,$$

which is the compatibility condition.  $\square$

**Lemma 8.7.** *Let  $u$  be a solution of  $(P_\epsilon)$ . Then if  $u \in C^{0,\alpha}(E_r)$  for some  $\alpha > (2s-1)_+$  and  $r > 1$ ,  $\partial_{Ae_n} u = 0$  from  $\Omega_1$  for  $|x'| < 1$  (in the sense of distributions).*

*Proof.* Identical to the proof of Lemma 7.6.  $\square$

**Theorem 8.8.** *Let  $u$  solve  $(P_\epsilon)$  with  $a_{s,i} \in \mathcal{L}_2 \cap \mathcal{L}_1^*$ .*

1. *Then  $u \in C^{0,\alpha}(E_1 \cap \bar{\Omega}_2)$  for every  $\alpha < \alpha_0$ ,  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_1)$  for every  $\alpha < \min\{\alpha_0 + 1 - 2s, 2 - 2s\}$ , and  $\partial_{Ae_n} u(x', 0^-) = 0$  for  $|x'| < 1$ .*
2. *Moreover, if  $\alpha_0 > 1$ , then  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_2)$  for each  $\alpha < \min\{\alpha_0 - 1, 2 - 2s\}$  and  $\partial_{(v_2,1)} u(x', 0) = 0$ .*
3. *If in addition  $a_{s,i}, A$  are compatible,  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_1)$  for every  $\alpha < \alpha_0 + 1 - 2s$  and  $u \in C^{1,\alpha}(E_1 \cap \bar{\Omega}_2)$  for each  $\alpha < \alpha_0 - 1$ .*

*The exponent  $\alpha_0$  depends only on  $a_{s,i}$  and, if in the case  $s = \frac{1}{2}$ , also on  $\langle \mathbf{b}, e_n \rangle$ .*

The proof of this theorem follows the bootstrapping argument of the previous section. We conclude with the following fact, which will be useful in subsequent sections:

**Proposition 8.9.** *Admissible solutions of  $(P_\epsilon)$  in the situation described above, provided  $a_{s,i} \in \mathcal{L}_2$ , are unique.*

*Proof.* We omit the  $\epsilon$  dependence for brevity. Let  $u$  and  $v$  be solutions; then  $w = u - v$  solves the equation with 0 right-hand side and 0 boundary data. In other words, for each  $\phi \in C_c^\infty(E_2)$  we have that

$$B_L[w, \phi] + B_N[w, \phi] = \int w \langle \mathbf{b}, \nabla \phi \rangle.$$

We claim that  $w \in W^{1,p}(E_2)$  for some  $p > 1$ . Indeed, we know that  $w$  is smooth away from  $\partial E_2 \cap \Gamma$ , and uniformly Hölder- $\alpha$  on  $\bar{E}_2$ . Applying the tangential regularity result, we see that  $|\partial_e w| \leq C$  on  $E_1$  for each  $e \perp e_n$ , and from interior estimates,  $|\partial_{e_n} w(x)| \leq C$  for  $x \in E_1 \cap \{|x_n| \geq \frac{1}{2}\}$ . Now for each  $x \in \Gamma \cap E_2$ , rescale to get

$$w_0(y) = w\left(\frac{y-x}{R}\right) - w(x).$$

Provided  $R > \frac{1}{2-|x'|}$ ,  $w_0$  solves  $(P_{1/R})$  on  $E_1$  and is bounded by  $R^{-\alpha}$  on  $E_2$ . Then we obtain that  $|\partial_e w_0| \leq C$  on  $E_1$  and that  $|\partial_{e_n} w_0(y)| \leq C$  on  $E_1 \cap \{|y_n| \geq \frac{1}{2}\}$ . The same can be done at  $x \in E_2 \setminus \Gamma$ , this time giving  $\partial_{e_n} w_0$  bounded on  $E_1$  provided  $R \geq \frac{1}{|y_n|}, \frac{1}{1-|y_n|}$ . Scaling back gives that  $|\partial_e w(x)| \leq C(2 - |x'|)^{\alpha-1}(1 - |x_n|)^{\alpha-1}$ , while  $|\partial_{e_n} w(x)| \leq C(2 - |x'|)^{\alpha-1}(1 - |x_n|)^{\alpha-1}|x_n|^{\alpha-1}$ . This guarantees that  $w \in W^{1,p}(E_2 \cap \Omega_2) \cap W^{1,p}(E_2 \cap \Omega_1)$  for

any  $p < \frac{1}{1-\alpha}$ , and from the continuity of  $w$  and the trace theorem  $w \in W^{1,p}(E_2)$ . We can then find a sequence  $\phi_l \rightarrow w$  strongly in  $H^1(\Omega_1) \cap H^s(\mathbb{R}^n) \cap W^{1,p}(E_2)$ . Using them as test functions and passing to the limit gives

$$B_L[w, w] + B_N[w, w] = \int w \langle \mathbf{b}, \nabla w \rangle = 0.$$

This implies  $w = 0$ , so  $u = v$ .  $\square$

*Remark 8.10.* In the cases of  $\alpha_0 < 1$  or  $\alpha_0 > 1$  and  $A, a_{s,i}$  compatible, it is possible to recover a transmission relation as in the previous section. The procedure is similar, using the next piece of the homogeneous solution as an improved barrier, but the construction is more challenging. Either the equation needs to be reduced to situation (H) with  $v_1, v_2 = 0$ , which is possible in this case, or the barriers need to account for tangential variation in  $u$ . The case of  $\alpha_0 = 1$  appears to require a secondary compatibility condition to admit a transmission relation: namely, that the numerator in the formula for  $v_2$  is zero.

## 9 Perturbative Theory

Now equipped with sufficiently powerful constant-coefficient estimates, we turn to the more general variable coefficient problem. The first section will discuss how the various quantities and conditions above behave under diffeomorphism and scaling. The rest will outline a perturbative framework for proving general regularity results near a boundary point.

### 9.1 Diffeomorphism Invariance

First consider a constant-coefficient equation of the type treated above, but with  $\Gamma$  not flat (we'll always assume  $\Gamma$  is at least locally  $C^{1,1}$ ). Up to a rotation that obviously preserves all of the quantities above, we may assume the plane  $\{x_n = 0\}$  is tangent to  $\Gamma$  at 0. Then the flattening map  $Q$  can be taken so that  $\nabla Q(0) = I$ ; as we will see this condition will preserve local ‘‘conformal’’ properties of  $a$  at 0. Moreover, because of the localization property,  $Q$  may be taken to be a global diffeomorphism of  $\mathbb{R}^n$  with global bounds on its derivatives (this is easily seen by replacing  $a, \Gamma$  outside a large ball with a flat extension and constant coefficients and then interpolating smoothly).

The transformed matrix  $\tilde{A} = (\nabla Q^T A \nabla Q) \circ Q^{-1}$  will be  $C^k$  in  $x$  provided  $\Gamma$  was  $C^{k+1}$ . Moreover,  $\tilde{A}(0) = A$ ; in particular, the conormal vector at 0 is the same. For the transformed drift,  $\tilde{\mathbf{b}}$  will be  $C^k$  if  $\Gamma$  is  $C^{k+1}$  as well, but will lose the divergence-free property. It will still be true, however, that  $\text{div} \tilde{\mathbf{b}} = \langle F, \mathbf{b} \rangle$  for some vector field  $F$  which is obtained from second derivatives of  $Q$  (so for instance if  $Q \in C^{1,1}$ , this is a bounded function). We will also make use of the fact that  $\tilde{\mathbf{b}}(0) = \mathbf{b}(0)$  when discussing optimal regularity in the case  $s = \frac{1}{2}$ .

Next, take the transformed form  $\tilde{a}(x, y) = a(Q^{-1}x, Q^{-1}y) \left( \frac{|x-y|}{|Q^{-1}x - Q^{-1}y|} \right)^{n+2s}$ . The translation regularity of  $\tilde{a}$  is similar to that of  $A$ , in the following sense: if  $a$  is in



$\mathcal{L}_k$  and  $\Gamma \in C^{k+1}$ , then  $\bar{a}$  will be  $C^{k-1,1}$  under the symmetric action  $z \mapsto \bar{a}(x+z, y+z)$  and at each point will be in  $\mathcal{L}_k$ .

More concretely, for any  $a$  we say that  $a$  is *decomposable* if  $a = a^1 1_{\{\Omega_2 \times \Omega_2\}} + a^2 1_{\{\Omega_2 \times \Omega_1\}} + a^2 1_{\{\Omega_1 \times \Omega_2\}}$ , with  $a^1, a^2$  continuous on  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x=y\}$  and satisfy  $0 \leq a^i \leq \Lambda$ ,  $a^i \geq \lambda$ . (For the purposes of the equation there are many equivalent decompositions, but we ask that one is fixed; this will give automatic extensions of some of the nonlocal operators in question). Then associate the following to  $a$ :

$$\begin{cases} a_{s,1}(x, z) = \frac{a^1(x, x+z) + a^1(x, x-z)}{2} \\ a_{a,1}(x, z) = \frac{a^1(x, x+z) - a^1(x, x-z)}{2} \\ a_{s,2}(x, z) = \frac{a^2(x, x+z) + a^2(x, x-z)}{2} \\ a_{a,2}(x, z) = \frac{a^2(x, x+z) - a^2(x, x-z)}{2} \end{cases}.$$

Thus  $a_{s,i}(x, z)$  is symmetric in  $z$ , and will play the role of the coefficients frozen at  $x$ , while  $a_{a,i}(x, z)$  is an antisymmetric remainder, which will exhibit cancellation properties making it low-order. It can be checked that  $a_{a,i} \equiv 0$  is equivalent to translation invariance of  $a^1$ .

We say  $a \in \mathcal{L}_k$  if  $a_{a,1}(x, \cdot), a_{s,i}(x, \cdot) \in \mathcal{L}_k$  for each  $x$ . On the other hand, we say  $a \in C_t^{k,\alpha}$  if

$$\sup_z |D_x^\beta a^i(x, x+z) - D_y^\beta a^i(y, y+z)| \leq C|x-y|^\alpha$$

for each multi-index  $|\beta| \leq k$ . The appropriate regularity notion for  $a_{a,1}$  is the following vanishing condition:  $a_{a,1} \in \mathcal{A}_\alpha$  if

$$\sup_x |a_{a,1}(x, z)| \leq C|z|^\alpha.$$

**Proposition 9.1.** *Let  $a, \bar{a}$ , and  $Q$  be as above, and assume  $a$  is decomposable. Then:*

1.  $\bar{a}$  is also decomposable.
2. If  $a \in \mathcal{L}_k$  and  $Q$  is uniformly  $C^k$ , then  $\bar{a} \in \mathcal{L}_k$ .
3. If  $a \in C_t^{k,\alpha}$ , then so is  $\bar{a}$  provided  $a \in \mathcal{L}_{k+1}$  and  $Q$  is  $C^{k+1,\alpha}$ .
4. If  $a_{s,i}(x, \cdot) \in \mathcal{L}_1^*$ , then so is  $\bar{a}_{s,i}(Qx, \cdot)$  provided  $a \in \mathcal{L}_1$  and  $Q \in C^{1,1}$ .
5. Moreover, if  $a$  is as in (4),  $\bar{a}_{s,i}^{(0)}(0, \cdot) = a_{s,i}^{(0)}(0, \cdot)$ . If  $a_{s,i}^{(0)}(x, z)$  is  $C^{0,\alpha}$  in both parameters on  $B_1 \times S^{n-1}$ , then for  $\delta < \delta_0$ ,

$$\sup_{|x| < \delta, z \in S^{n-1}} |\bar{a}_{s,i}^{(0)}(x, z) - a_{s,i}^{(0)}(0, z)| \leq C\delta^\alpha$$

6. If  $a_{a,i} \in \mathcal{A}_\alpha$ , then so is  $\bar{a}_{a,i}$  provided  $Q \in C^{1,\alpha}$  and  $a \in \mathcal{L}_1$ .

*Proof.* For (1), this is obvious from the expression for  $\bar{a}$  and the fact that  $Q$  maps  $\Gamma \rightarrow \Gamma$ .

For (2) it suffices to check that  $|D_z^\beta \tilde{a}^i(x, x+z)| \leq C|z|^{-k}$  for  $|\beta| = k$ . Computing first  $|D_z^\beta[a^i(Q^{-1}x, Q^{-1}(x+z))]|$  gives

$$\begin{aligned} |D_z^\beta[a^i(Q^{-1}x, Q^{-1}(x+z))]| &\leq C \sum_{l=1}^k |D^l a^i(Q^{-1}x, Q^{-1}(x+z))| |D^{k-l+1} Q^{-1}(x+z)|^l \\ &\leq C \sum_{l=1}^k \frac{1}{|Q^{-1}x - Q^{-1}(x+z)|^l} \leq C|z|^{-k}. \end{aligned}$$

The other factor depends only on  $Q$  :

$$\begin{aligned} \left| D_z^\beta \left( \frac{|z|}{|Q^{-1}x - Q^{-1}(x+z)|} \right)^{n+2s} \right| &\leq C \sum_{l=0}^k |D^l|z|^{n+2s}| |D^{k-l}|Q^{-1}x - Q^{-1}(x+z)|^{-n-2s}| \\ &\leq C \sum_{l=0}^k |z|^{n+2s-l} \frac{C(|D^{k-l}Q^{-1}|)}{|z|^{n+2s+k-l}} \\ &\leq C|z|^{-k}. \end{aligned}$$

Then the conclusion follows from Leibniz rule.

For (3), we only check the case  $k = 0$ ; for higher  $k$  the computation is similar but somewhat more tedious, and will not be needed below. First,

$$\begin{aligned} |a(Q^{-1}x, Q^{-1}(x+z)) - a(Q^{-1}y, Q^{-1}(y+z))| &\leq C|Q^{-1}x - Q^{-1}y|^\alpha \\ &\quad + |a(Q^{-1}y, Q^{-1}y - Q^{-1}x + Q^{-1}(x+z)) - a(Q^{-1}y, Q^{-1}(y+z))| \\ &\leq C|x - y|^\alpha + C|z|^{-1} |Q^{-1}y - Q^{-1}x + Q^{-1}(x+z) - Q^{-1}(y+z)| = S. \end{aligned}$$

If  $|z| \leq |x - y|$ , then

$$\begin{aligned} S &\leq C|x - y|^\alpha + C|z|^{-1} (|\nabla Q^{-1}(x)z - \nabla Q^{-1}(y)z| + C|z|^{1+\alpha}) \\ &\leq C|x - y|^\alpha + C(|x - y|^\alpha + |z|^\alpha) \\ &\leq C|x - y|^\alpha, \end{aligned}$$

while if not,

$$\begin{aligned} S &\leq C|x - y|^\alpha + C|z|^{-1} (|\nabla Q^{-1}(x)(y - x) - \nabla Q^{-1}(x+z)(y - x)| + C|x - y|^{1+\alpha}) \\ &\leq C|x - y|^\alpha + C|z|^{-1} (|x - y||z|^\alpha + C|x - y|^{1+\alpha}) \\ &\leq C|x - y|^\alpha. \end{aligned}$$

The other factor can easily be seen to be Hölder- $\alpha$ , and the conclusion then follows from the algebra property of Hölder spaces.

For (4, 5), we use the notation  $y = Q^{-1}x$ :

$$\begin{aligned}
\bar{a}_{s,1}(x, z) &= \frac{\bar{a}^1(x, x+z) + \bar{a}^1(x, x-z)}{2} \\
&= \frac{1}{2} \left[ \frac{a^1(Q^{-1}x, Q^{-1}(x+z))|z|^{n+2s}}{|Q^{-1}(x+z)|^{n+2s}} + \frac{a^1(Q^{-1}x, Q^{-1}(x-z))|z|^{n+2s}}{|Q^{-1}(x-z)|^{n+2s}} \right] \\
&= \frac{1}{2|\nabla Q^{-1}(y)\hat{z}|} \left[ a^1(y, y + \nabla Q^{-1}(y)z + O(|z|^2)) \right. \\
&\quad \left. + a^1(y, y - \nabla Q^{-1}(y)z + O(|z|^2)) \right] + O(|z|) \\
&= \frac{1}{|\nabla Q^{-1}(y)\hat{z}|} a_{s,1}(y, \nabla Q^{-1}(y)z) + O(|z|^2)/|z| + O(|z|) \\
&= \frac{1}{|\nabla Q^{-1}(y)\hat{z}|} a_{s,1}^{(0)}(y, \nabla Q^{-1}(y)z) + O(|z| + \omega(|z|)),
\end{aligned}$$

which immediately shows that  $\bar{a}_{s,1} \in \mathcal{L}_1^*$ . (5) now follows from the fact that  $\nabla Q^{-1}(0) = I$  and the formula above. For  $i = 2$  the computation is the same.

Finally, (6) follows from a similar argument.

$$\begin{aligned}
\bar{a}_{a,1}(x, z) &= \frac{\bar{a}^1(x, x+z) - \bar{a}^1(x, x-z)}{2} \\
&= \frac{1}{2} \left[ \frac{a^1(y, Q^{-1}(x+z))|z|^{n+2s}}{|y - Q^{-1}(x+z)|^{n+2s}} - \frac{a^1(y, Q^{-1}(x-z))|z|^{n+2s}}{|y - Q^{-1}(x-z)|^{n+2s}} \right] \\
&= \frac{1}{2} \left[ \frac{a^1(y, y + \nabla Q^{-1}(y)z + O(|z|^{1+\alpha}))}{|\nabla Q^{-1}(y)|^{n+2s}} - \frac{a^1(y, y - \nabla Q^{-1}(y)z + O(|z|^{1+\alpha}))}{|\nabla Q^{-1}(y)|^{n+2s}} \right] \\
&\quad + O(|z|^\alpha) \\
&= \frac{1}{2|\nabla Q^{-1}(y)|^{n+2s}} \left[ a^1(y, y + \nabla Q^{-1}(y)z) - a^1(y, y - \nabla Q^{-1}(y)z) \right] + O(|z|^\alpha) \\
&= \frac{1}{|\nabla Q^{-1}(y)|^{n+2s}} a_{a,1}(y, \nabla Q^{-1}(y)z) + O(|z|^\alpha) \\
&= O(|z|^\alpha).
\end{aligned}$$

□

A consequence of this proposition is that if we start with a translation invariant kernel in  $\mathcal{L}_2 \cap \mathcal{L}_1^*$  and  $\Gamma \in C^{1,1}$ , we obtain a kernel on a flat interface which is no longer translation invariant, but is in  $C_t^{0,1} \cap \mathcal{L}_2 \cap \mathcal{A}_1$ , has symmetric part in  $\mathcal{L}_1^*$  at the origin, and preserves the limiting homogeneous structure there (and as a consequence properties such as the values of  $A_{s,i}$ ,  $M_{s,i}$ ,  $\alpha_0$ , and compatibility.) If, furthermore,  $a_{s,i}^{(0)}(z)$  is smooth enough, then  $\bar{a}_{s,i}(x, z)$  will be close to  $a_{s,i}^{(0)}$  for  $x$  small.

## 9.2 Scaling Properties

Now we discuss how the flattened problem behaves under dilations centered at the origin. Consider the problem  $(P_\epsilon)$  satisfied by  $u_\epsilon(x) = u(\epsilon x)$ . The transformed matrix

$A^\epsilon(x) = A(\epsilon x)$  satisfies  $A^\epsilon(0) = A(0)$ , and also if  $A$  had the modulus of continuity  $\omega_2$ , then

$$\text{osc}_{E_1} A^\epsilon \leq \omega_2(\epsilon).$$

The same statements hold for the vector field  $\mathbf{b}^\epsilon$  and right-hand side  $f^\epsilon$ , and  $\text{div} \mathbf{b}^\epsilon = \epsilon \text{div} \mathbf{b}$ . For the nonlocal kernel  $a^\epsilon$ , we have that if  $a \in \mathcal{L}_k$ , then so is  $a^\epsilon$  with the same constant, for

$$|D_z^\beta a^{\epsilon,i}(x, x+z)| \leq \epsilon^{|\beta|} |D_z^\beta a^i(\epsilon x, \epsilon(x+z))| \leq C|z|^{-|\beta|}.$$

If  $a \in C_t^{k,\alpha}$ , then we have that

$$\sup_z |D_x^\beta a^{\epsilon,i}(x, x+z) - D_y^\beta a^{\epsilon,i}(y, y+z)| \leq C\epsilon^{|\beta|+\alpha} |x-y|^\alpha,$$

while if  $a_{a,i} \in \mathcal{A}_\alpha$ , then

$$|a_{a,i}^\epsilon(x, z)| = |a_{a,i}(\epsilon x, \epsilon z)| \leq C\epsilon^\alpha |z|^\alpha.$$

If  $a_{s,i}(0, \cdot) \in \mathcal{L}_1^*$ , then  $a_{s,i}^\epsilon(0, z)$  satisfies

$$a_{s,i}^\epsilon(0, z) = a_{s,i}(0, \epsilon z) = a_{s,i}^{(0)}(0, z) + \omega(\epsilon|z|),$$

so  $a_{s,i}^\epsilon(0, \cdot) \in \mathcal{L}_1^*$  and has the same homogeneous part as  $a_{s,i}$ .

### 9.3 An Approximation Lemma

The main ingredient in the perturbative theorem is the following lemma about approximation by translation-invariant equations. We will give two frameworks for the perturbative theory. The first will use the method of Campanato, and is relatively straightforward. However, it seems to lack the flexibility to improve regularity to the near-optimal level in certain  $(\alpha_0, s)$  ranges. The second is a classical improvement of flatness argument, incorporating a substantially more sophisticated approximation lemma. While the setup is more complicated, the conclusions are stronger.

Below,  $x_0$  may lie outside of  $\Gamma$ . It is helpful to introduce the following: let  $\alpha_0(x)$  be the optimal regularity for the equation with coefficients frozen at  $x$  (i.e. the exponent in Theorem 8.8). Then let

$$\alpha_0(\Omega) = \inf_{x \in \Omega} \alpha_0(x).$$

Also, in this section we will find it useful to introduce the following *generalized problem*. Let  $f_1$  be an  $L^2$  vector field on  $\Omega_1$  and  $h(x, y)$  a (not necessarily symmetric) function satisfying

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|h(x, y)|^2 dx dy}{|x - y|^{n+2s}} < \infty.$$

Then  $u$  solves the generalized problem (P) on  $\Omega$  if for each  $\phi \in C_c^\infty(\mathbb{R}^n)$ :

$$\begin{aligned} B_L[u, \phi] + B_N[u, \phi] &= \int u \langle \mathbf{b}, \nabla \phi \rangle + f \phi \\ &+ \int_{\Omega_1} \langle f_1, \nabla \phi \rangle + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{h(x, y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

We remark that we will never solve the generalized problem; it is simply a way of keeping track of some extra terms on the right-hand side that come out of the bootstrap argument. The following proposition is a basic fact about fractional Sobolev spaces; we include the elementary proof for completeness.

**Proposition 9.2.** *Let  $u \in H^s(\mathbb{R}^n)$ . Then there is a constant  $C = C(n, s)$  such that*

$$\int_{\mathbb{R}^n} \|u(x', \cdot)\|_{H^s(\mathbb{R})}^2 dx' \leq C \|u\|_{H^s(\mathbb{R}^n)}^2.$$

*Proof.* Observe it suffices by density to prove this for smooth compactly supported functions. We use the Fourier transform formulation of fractional Sobolev spaces and the Plancharel theorem:

$$\begin{aligned} \int_{\mathbb{R}^n} \|u(x', \cdot)\|_{H^s(\mathbb{R})}^2 dx' &= C \int_{\mathbb{R}^n} (1 + |\xi_n|^2)^{s/2} \hat{u}(\xi) d\xi \\ &\leq C \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2} \hat{u}(\xi) d\xi \\ &= C \|u\|_{H^s(\mathbb{R}^n)}^2. \end{aligned}$$

Here  $\hat{u}$  denoted the Fourier transform of  $u$ , and the constant comes only from the normalization used in the definition of  $H^s$ .  $\square$

The following lemma is the first half of the approximation devised in this section. We will use the notation

$$\mathcal{D}(x, y, U, \alpha) = (d(x, U^c)^{\alpha-1} + d(y, U^c)^{\alpha-1}), \quad (9.1)$$

where  $U$  is an open set and  $\alpha$  a real number, in some weighted estimates below.

**Lemma 9.3.** *Let  $u$  be an admissible solution of the generalized problem for  $(P_\epsilon)$  (with data  $A, a, \mathbf{b}, f, f_1, h$ ) on  $B_3(x_0)$ , with  $\|u\|_{C^{0,\alpha'}(\mathbb{R}^n)} \leq 1$ . Assume  $a \in \mathcal{L}_2 \cap \mathcal{L}_1^*$  with  $\alpha_0(\Omega) > (2s-1)_+$ . Let  $u_0$  be the unique solution to the following Dirichlet problem:*

$$\left\{ \begin{array}{l} \forall \phi \in C_c^\infty(B_2(x_0)), \quad 0 = \int_{\Omega_1} \langle A^\epsilon(x_0) \nabla u_0, \nabla \phi \rangle + \\ \quad + \epsilon^{2(1-s)} \int_{\Omega_2} \int_{\Omega_1} \frac{[u_0(x) - u_0(y)] a_{s,2}^\epsilon(x_0, x-y) [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ \quad + \epsilon^{2(1-s)} \int_{\Omega_2} \int_{\Omega_2} \frac{a_{s,1}^\epsilon(x_0, x-y) [u_0(x) - u_0(y)] [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ \quad - \int \epsilon u_0 \langle \mathbf{b}(x_0), \nabla \phi \rangle + \epsilon^2 \left[ 1_{\Omega_1} \oint_{\Omega_1 \cap B_2(x_0)} f + 1_{\Omega_2} \oint_{\Omega_2 \cap B_2(x_0)} f \right] \phi \\ \forall x \in B_2^c(x_0), \quad u_0(x) = u(x) \end{array} \right. \quad (9.2)$$

*Assume the following hold for some parameters  $\beta \geq 0, \eta > 0, 0 < \alpha < \alpha_0(x_0) - (2s-1)_+$ , small enough, and  $x \in B_2(x_0)$ :*

1.  $|A^\epsilon(x) - A^\epsilon(x_0)| \leq \eta|x - x_0|^\beta$
2.  $\epsilon^{2s-1}|\mathbf{b}^\epsilon(x) - \mathbf{b}^\epsilon(x_0)| \leq \eta|x - x_0|^{(\beta+1-2s)_+}$
3.  $\epsilon^{2s-1} \sup_{B_2(x_0)} |\operatorname{div} \mathbf{b}^\epsilon| \leq \eta$
4.  $\epsilon^{2s} [\operatorname{osc}_{B_r(x_0) \cap \Omega_1} f^\epsilon + \operatorname{osc}_{B_r(x_0) \cap \Omega_2} f^\epsilon] \leq \eta r^{(\beta-2s)_+}$  for  $r \leq 2$
5.  $\sup_{z \in \mathbb{R}^n} |a^{\epsilon,i}(x, x+z) - a^{\epsilon,i}(y, y+z)| \leq \eta|x - y|^\beta$  for  $x, y \in B_3(x_0)$
6.  $\sup_{x, z \in \mathbb{R}^n} |a_{a,i}^\epsilon(x, z)| \leq \eta|z|^\beta$

Then  $v = u - u_0$  is an (admissible)  $C^{0,\alpha'}$  solution to the generalized problem for  $(P_\epsilon)$  on  $B_2(x_0)$  with data  $A, a, \mathbf{b}, \tilde{f}, \tilde{f}_1, \tilde{h}$  which satisfy the following:

1.  $\epsilon^{2s} |\tilde{f}^\epsilon(x)| \leq C\eta|x - x_0|^{(\beta-2s)_+} d(x, B_2^c(x_0))^{\alpha-1}$
2.  $\epsilon |\tilde{f}_1^\epsilon(x)| \leq \epsilon |f_1^\epsilon(x)| + C\eta|x - x_0|^\beta d(x, B_2^c(x_0))^{\alpha-1}$
3. If  $|x - y| < \frac{1}{2}|x_n|$ ,
$$|\tilde{h}^\epsilon(x, y)| + |\tilde{h}^\epsilon(y, x)| \leq [|h^\epsilon(x, y)| + |h^\epsilon(y, x)|] \\ + C\eta(|x - y| \wedge 1)|x_n|^{\alpha+(2s-1)_+-1} (|x - x_0|^\beta + |y - x_0|^\beta) \mathcal{D}(x, y, B_2(x_0), \alpha)$$
4.  $|\tilde{h}^\epsilon(x, y)| \leq |h^\epsilon(x, y)| + C\eta(|x - y| \wedge 1)^{\alpha+(2s-1)_+} (|x - x_0|^\beta + |y - y_0|^\beta) \mathcal{D}(x, y, B_2(x_0), \alpha)$
5.  $\epsilon^{2s} \|\tilde{f}\|_{L^2(B_2(x_0))} + \epsilon \|\tilde{f}_1 - f_1\|_{L^2(B_2(x_0))} + \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\tilde{h}(x, y) - h(x, y)|^2}{|x - y|^{n+2s}} \right)^{1/2} \leq C\eta.$

For the rest of this section, the case  $\beta = 0$  would be sufficient, but when  $\beta > 0$  the above gives improved scaling for the generalized right-hand sides, which will be helpful later. A special but useful case is if the original problem has  $f_1, h = 0$ , in which case the difference between  $u$  and  $u_0$  still only solves a generalized problem. The conclusion should be interpreted as saying that whenever the coefficients of the equation have small oscillation,  $v$  solves an equation with small right-hand side.

*Proof.* We ignore the topic of admissibility; this is easily justified using the uniqueness of  $u_0$ . We have the energy estimates

$$\|u_0\|_{H^1(B_4(x_0) \cap \Omega_1) \cap H^s(B_4(x_0))} \leq C\|u\|_{H^1(B_4(x_0) \cap \Omega_1) \cap H^s(B_4(x_0))} \leq C,$$

with the constant independent of  $\epsilon$ . Set  $U = B_2(x_0)$ , let  $\phi \in C_c^\infty(U)$ , and use the notation  $B_L^0, B_N^0$  for the quantities in (9.2). Then

$$\begin{aligned} B_L^\epsilon[v, \phi] &= B_L^\epsilon[u, \phi] - \int_{\Omega_1} \langle A^\epsilon \nabla u_0, \nabla \phi \rangle \\ &= B_L^\epsilon[u, \phi] - B_L^0[u_0, \phi] + \int_{\Omega_1} \langle (A^\epsilon(x_0) - A^\epsilon) \nabla u_0, \nabla \phi \rangle \\ &= B_L^\epsilon[u, \phi] - B_L^0[u_0, \phi] + \int_{\Omega_1} \langle K_0, \nabla \phi \rangle. \end{aligned}$$

Notice that  $\|K_0\|_{L^2(U)} \leq C\eta$  from the energy estimate above, while from the constant coefficient estimate (appropriately scaled) of Section 8, we have that  $|\nabla u_0(x)| \leq Cd(x, \partial U)^{\alpha-1}$  for  $x \in \Omega_2$ , which implies  $|K_0| \leq C\eta|x - x_0|^\beta d(x, \partial U)^{\alpha-1}$ .

For the drift,

$$\begin{aligned} \int \langle \mathbf{b}^\epsilon, \nabla \phi \rangle v &= \int \langle \mathbf{b}^\epsilon, \nabla \phi \rangle u - \langle \mathbf{b}^\epsilon(x_0), \nabla \phi \rangle u_0 - \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla \phi \rangle u_0 \\ &= \int \langle \mathbf{b}^\epsilon, \nabla \phi \rangle u - \langle \mathbf{b}^\epsilon(x_0), \nabla \phi \rangle u_0 + \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle \phi + K_1 \phi \end{aligned}$$

where  $|K_1| \leq C\eta/\epsilon^{2s-1}$  from the assumption on the divergence of  $\mathbf{b}^\epsilon$ . We used that  $u_0 \in W^{1,p}(U)$  for some  $p > 1$ , which follows from Lemma 8.2; indeed,  $|\nabla u_0| \leq Cd(x, \Gamma \cup \partial U)^{\alpha-1}$  (see also the proof of Proposition 8.9). It is now convenient to solve the following Dirichlet problem for the fractional Laplacian:

$$\begin{cases} (-\Delta)^{1/2} w(x) = 1_U(x) \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle & x \in \Omega \\ w(x) = 0 & x \notin \Omega \end{cases} \quad (9.3)$$

where  $\Omega$  is some smooth domain with  $U \subset \subset \Omega$ . The relevant fact about the right-hand side is that it is controlled by  $C\eta|x - x_0|^\beta/\epsilon^{2s-1}[d(x, \Gamma)^{\alpha+(2s-1)_+-1} + d(x, U^c)^{\alpha-1}]$ . The following claim can be easily derived from scaling:

**Claim.** *There exists a unique finite-energy solution  $w$  to (9.3), with  $\|w\|_{H^{1/2}} \leq C\eta/\epsilon^{2s-1}$ . Moreover,  $w$  is Hölder continuous on  $U$  with*

$$|w(x) - w(y)| \leq C\epsilon^{1-2s}\eta|x - y|^{\alpha+(2s-1)_+} \left( |x - x_0|^\beta + |y - x_0|^\beta \right) \mathcal{D}(x, y, U, \alpha)$$

and

$$|w(x) - w(y)| \leq C\epsilon^{1-2s}\eta|x - y||x_n|^{\alpha+(2s-1)_+-1} \left( |x - x_0|^\beta + |y - x_0|^\beta \right) \mathcal{D}(x, y, U, \alpha)$$

provided  $|x - y| < \frac{1}{2}|x_n|$ .

*Proof.* We claim that the right-hand side in (9.3) lies in the dual space of  $H_0^{1/2}(\Omega)$  (by which we mean the closure of  $C_c^\infty(\Omega)$  in  $H^{1/2}(\mathbb{R}^n)$ ), in the sense that

$$\left| \int_{\mathbb{R}^n} 1_U(x) \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle w \right| \leq C\|w\|_{H^{1/2}}$$

for every  $w \in C_0^\infty(\Omega)$  with some uniform constant  $C$ . Indeed, this is a consequence of Proposition 9.2 and Sobolev embedding. For  $w$  supported away from  $\partial U$ , we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} 1_U(x) \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle w \right| &\leq C\eta\epsilon^{1-2s} \int |x_n|^{\alpha_0-1} 1_U(x', x_n) |w(x', x_n)| dx_n dx' \\ &\leq C\eta\epsilon^{1-2s} \int \sup_{x_n} |w(x', x_n)| dx' \\ &\leq C\eta\epsilon^{1-2s} \left( \int \|w(x', \cdot)\|_{H^{1/2}(\mathbb{R})}^2 \right)^{1/2} \\ &\leq C\eta\epsilon^{1-2s} \|w\|_{H^{1/2}(\mathbb{R}^n)}. \end{aligned}$$

A similar computation works for  $w$  supported near  $\partial U$ . It then follows immediately from Lax-Milgram theorem that there is a unique finite-energy solution to (9.3), and that it satisfies the energy estimate as promised. For the point estimates, we will first apply Campanato criterion. We have that:

$$\begin{aligned} \oint_{B_r(z)} |w(x) - \oint_{B_r(z)} w| dx &\leq Cr^{1-n} \int_{B_{2r}(z)} |(-\Delta)^{1/2} w| \\ &= Cr^{1-n} \int_{B_{2r}(z) \cap U} |\langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle| \\ &\leq C\eta\epsilon^{1-2s} |z - x_0|^\beta d(z, U^c)^{-1} r^{\alpha+(2s-1)_+}, \end{aligned}$$

which implies the first estimate. The second follows from this, interior estimates, and scaling.  $\square$

Now the error term from the drift can be re-expressed as follows:

$$\begin{aligned} \int \langle \mathbf{b}^\epsilon - \mathbf{b}^\epsilon(x_0), \nabla u_0 \rangle \phi &= \int \phi (-\Delta)^{1/2} w \\ &= c \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[w(x) - w(y)][\phi(x) - \phi(y)]}{|x - y|^{n+1}} \\ &= \epsilon^{2s-1} I_1[\phi]. \end{aligned}$$

The nonlocal terms can be simplified (first doing the  $\Omega_1 \times \Omega_2$  ones):

$$\begin{aligned} &\int_{\Omega_2} \int_{\Omega_1} \frac{[v(x) - v(y)]a^\epsilon(x, y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &= \int_{\Omega_2} \int_{\Omega_1} \frac{[u(x) - u(y)]a^\epsilon(x, y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &\quad - \int_{\Omega_2} \int_{\Omega_1} \frac{[u_0(x) - u_0(y)]a_{s,2}^\epsilon(x_0, x - y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &\quad + \int_{\Omega_2} \int_{\Omega_1} \frac{[u_0(x) - u_0(y)](a_{s,2}^\epsilon(x_0, x - y) - a_{s,2}^\epsilon(x, x - y))[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &\quad + \int_{\Omega_2} \int_{\Omega_1} \frac{[u_0(x) - u_0(y)]a_{a,2}^\epsilon(x, x - y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \end{aligned}$$

where we have used that  $a^2(x, y) = a_{s,2}(x, x - y) - a_{a,2}(x, x - y)$ . The first two terms are parts of  $B_N^\epsilon$  and  $B_N^0$  respectively; the others, which we denote by  $I_2[\phi]$ , are of the form

$$I_2[\phi] = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{K_2(x, y)[u_0(x) - u_0(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dx dy,$$

where

$$|K_2(x, y)(u_0(x) - u_0(y))| \leq \eta C |x - y|^{\alpha+(2s-1)_+} \mathcal{D}(x, y, U, \alpha)[|x - x_0|^\beta + |y - x_0|^\beta].$$



Similarly the nonlocal terms over  $\Omega_2 \times \Omega_2$  can be written as

$$\begin{aligned} & \int_{\Omega_2} \int_{\Omega_2} \frac{a^\epsilon(x, y)[w(x) - w(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &= \int_{\Omega_2} \int_{\Omega_2} \frac{a^\epsilon(x, y)[u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &- \int_{\Omega_2} \int_{\Omega_2} \frac{a_{s,1}^\epsilon(x_0, x - y)[u_0(x) - u_0(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dy dx \\ &+ I_3[\phi] \end{aligned}$$

with  $I_3$  satisfying the same properties as  $I_2$ .

Now by subtracting the equations for  $u_0$  and  $u$ , we have the following equation for  $v$ :

$$\begin{aligned} 0 = & -B_L^\epsilon[v, \phi] - \epsilon^{2(1-s)} B_N^\epsilon[v, \phi] + \epsilon \int v \langle b^\epsilon, \nabla \phi \rangle \\ & + \int_{\Omega_1} \langle K_0, \nabla \phi \rangle + \epsilon \int K_1 \phi + \epsilon I_1[\phi] \\ & + \epsilon^{2(1-s)} (I_2[\phi] + I_3[\phi]) - \epsilon^2 \int \left[ f^\epsilon - \left( 1_{\Omega_1} \int_{\Omega_1 \cap B_2(x_0)} f + 1_{\Omega_2} \int_{\Omega_2 \cap B_2(x_0)} f \right) \right] \phi \\ & + \epsilon^{2(1-s)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{h^\epsilon(x, y)[\phi(x) - \phi(y)] dx dy}{|x - y|^{n+2s}} + \epsilon \int_{\Omega_1} \langle f_1^\epsilon, \nabla \phi \rangle. \end{aligned} \quad (9.4)$$

It is now easy to see from the estimates we have shown that this generalized problem satisfies the conclusions of the lemma.  $\square$

This lemma is meant to be partnered with the following one, which states that solutions to a generalized problem with small data are small, both in energy and (under some stronger structural assumptions) in  $L^\infty$ . The fact that the estimates above improve as  $x$  gets closer to  $x_0$  will not be relevant to the approximation below; rather, that will only be used for some scaling arguments in later sections.

**Lemma 9.4.** *Let  $u$  be an admissible solution of the generalized problem for  $(P_\epsilon)$  on  $B_2(x_0)$ , with  $\|u\|_{C^{0,\alpha'}(\mathbb{R}^n)} \leq 1$  and  $u$  supported on  $B_2(x_0)$ . Then there is a constant  $C_1$ , independent of  $\epsilon$ , such that if*

$$\epsilon^{2s} \|f\|_{L^2(B_2(x_0))} + \epsilon \|f_1\|_{L^2(B_2(x_0))} + \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|h(x, y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2} + \epsilon^{2s-1} \sup_{B_2(x_0)} |\operatorname{div} b^\epsilon| \leq \eta,$$

then

$$\|u\|_{H^1(\Omega_1) \cap H^s(\mathbb{R}^n)} \leq C_1 \eta. \quad (9.5)$$

Under the additional assumptions

1.  $\epsilon^{2s} |f^\epsilon(x)| \leq C \eta d(x, B_2^c(x_0))^{\alpha-1}$
2.  $\epsilon |f_1^\epsilon(x)| \leq C \eta d(x, B_2^c(x_0))^{\alpha-1}$

3. if  $|x - y| < \frac{1}{2}|x_n|$ , then
- $$|h^\epsilon(x, y)| + |h^\epsilon(y, x)| \leq C\eta(|x - y| \wedge 1)|x_n|^{\alpha + (2s-1)_+ - 1}\mathcal{D}(x, y, B_2(x_0), \alpha)$$
4.  $|h^\epsilon(x, y)| \leq C\eta(|x - y| \wedge 1)^\alpha \mathcal{D}(x, y, B_2(x_0), \alpha)$

we have this  $L^\infty$  estimate (for some universal  $\gamma > 0$ ).

$$\|u\|_{L^\infty(B_1(x_0))} \leq C_1\eta^\gamma. \quad (9.6)$$

*Proof.* This equation admits an energy inequality (this is justified by writing down the equation above for the approximate problem, setting  $\phi = u$ , noticing the drift term on the left vanishes up to a lower-order term, and passing to the limit): we have that

$$\begin{aligned} B_L^\epsilon[u, u] + \epsilon^{2(1-s)} B_N^\epsilon[u, u] &\leq \int_{\Omega_1} \epsilon \langle f_1^\epsilon, \nabla u \rangle + \int \epsilon^2 f u + u^2 \epsilon \operatorname{div} \mathbf{b}^\epsilon \\ &\quad + \epsilon^{2(1-s)} \int \frac{h^\epsilon(x, y)[u(x) - u(y)]}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

We bound each term:

$$\int_{\Omega_1} \epsilon \langle f_1^\epsilon, \nabla u \rangle \leq \mu \int_{\Omega_1} |\nabla u|^2 + C_\mu \epsilon^2 \|f_1\|_{L^2(B_2)}^2 \leq \mu \int_{\Omega_1} |\nabla u|^2 + C_\mu \eta^2,$$

where the first term is reabsorbed. Similarly,

$$\int (\epsilon^2 |f^\epsilon| + \epsilon \operatorname{div} \mathbf{b}^\epsilon u) u \leq \epsilon^{2(1-s)} \mu \int u^2 + C_\mu \eta^2.$$

That only leaves the nonlocal integral, which is treated as follows:

$$\begin{aligned} \epsilon^{2(1-s)} \int \frac{h^\epsilon(x, y)[u(x) - u(y)]}{|x - y|^{n+2s}} dx dy &\leq \epsilon^{2(1-s)} \left[ \mu \|u\|_{H^s(\mathbb{R}^n)}^2 + C_\mu \int \frac{|h^\epsilon(x, y)|^2}{|x - y|^{n+2s}} dx dy \right] \\ &\leq \mu \epsilon^{2(1-s)} \|u\|_{H^s(\mathbb{R}^n)}^2 + C_\mu \eta^2 \end{aligned}$$

This gives that

$$B_L^\epsilon[u, u] + \epsilon^{2(1-s)} B_N^\epsilon[u, u] \leq C\eta^2.$$

A second energy inequality can be obtained by using  $u - R[u]$  as a test function, where  $R$  is the reflection operator across  $\Gamma$ . We omit the details, but it is easily verified that

$$B_N^\epsilon[u - R[u], u - R[u]] \leq C\eta^2 + C\|R[u]\|_{H^1(\mathbb{R}^n)}^2 \leq C\eta^2.$$

This implies (9.5).

For the  $L^\infty$  conclusion, we need a level set energy inequality and a somewhat more subtle approach to controlling the nonlocal terms, this time fully utilizing the power of the constant-coefficient estimate. An estimate with sharp dependence on  $\eta$  would require some extra examination of regularity of  $u$  up to the boundary  $\partial B_2$ , which we do not wish to pursue. Thus we content ourselves with the following well-known argument.

Recall that  $\|u\|_{C^{0,\alpha'}(\mathbb{R}^n)} \leq 1$ . As  $u$  vanishes at  $\partial B_2(x_0)$ , this means that

$$l(r) = \sup_{B_r^c(x_0)} |u(x)| \leq C(2-r)^{\alpha'}.$$

Set  $l_k = l(r) + C_2(1 - 2^{-k})$ , where  $C_2$  will be chosen below. Now use  $(u - l_k)_+$  as a test function in (9.4) and estimate the terms on the right as follows: since by our improved estimate  $f_1$  is bounded, we have

$$\int_{\Omega_1} \langle \epsilon f_1^\epsilon, \nabla(u - l_k)_+ \rangle \leq \mu \|(u - l_k)_+\|_{H^1(\Omega_1)}^2 + C_\mu \eta^2 (2-r)^{-2} |\{u > l_k\}|.$$

Also, the remaining local terms are straightforward to bound:

$$\int (\epsilon^2 |f^\epsilon| + \epsilon \operatorname{div} \mathbf{b}^\epsilon (u - l_k)_+) (u - l_k)_+ \leq \mu \epsilon^{2(1-s)} \int (u - l_k)_+^2 + C \eta^2 |\{u > l_k\}|,$$

where the first term can be reabsorbed. This leaves only the nonlocal term:

$$\begin{aligned} & \epsilon^{2(1-s)} \int \frac{h^\epsilon(x, y) [(u - l_k)_+(x) - (u - l_k)_+(y)]}{|x - y|^{n+2s}} \\ & \leq \mu \epsilon^{2(1-s)} \|(u - l_k)_+\|_{H^s}^2 + C \epsilon^{2(1-s)} \int_{\{u > l_k\}} \int_{\mathbb{R}^n} \frac{h^\epsilon(x, y)^2 + h^\epsilon(y, x)^2}{|x - y|^{n+2s}} dy dx \\ & \leq C \eta^2 (2-r)^{-2} \int_{\{u > l_k\}} \int_{\mathbb{R}^n} \frac{(|x - y| \wedge 1)^{2(s+\alpha/2)} |x_n|^\gamma d(y, \partial B_2(x_0))^{\alpha-1}}{|x - y|^{n+2s}} dy dx \\ & \quad + \mu \epsilon^{2(1-s)} \|(u - l_k)_+\|_{H^s}^2, \end{aligned}$$

where  $\gamma = 2(\alpha/2 + (2s-1)_+ - s) > -1 + (2s-1)_+ \geq -1$ . We have used the hypotheses (3, 4) and the fact that  $(u - l_k)_+$  is supported on  $B_r$ . The second piece is reabsorbed, while for the first,

$$\begin{aligned} & \leq C \eta^2 (2-r)^{-2} \int_{\{u > l_k\}} |x_n|^\gamma dx \\ & \leq C \eta^2 (2-r)^{-2} \int \int_0^{|\{u(x', \cdot) > l_k\}|_1} |x_n|^\gamma dx_n dx' \\ & \leq C \eta^2 (2-r)^{-2} \int |\{u(x', \cdot) > l_k\}|_1^{1+\gamma} dx'. \end{aligned}$$

The subscript  $|\cdot|_1$  means one-dimensional (Hausdorff) measure. As we will see shortly, this unusual asymmetric bound is compatible with Sobolev embedding, and this is the key point of the estimate. Set

$$\mathcal{A}_k = \int (u - l_k)_+^2 + |\{u > l_k\}| + \int |\{u(x', \cdot) > l_k\}|_1^{1+\gamma} dx';$$

we have just shown that

$$\|(u - l_k)_+\|_{H^1(\Omega_1)}^2 + \epsilon^{2(1-s)} \|(u - l_k)_+\|_{H^s(\mathbb{R}^n)}^2 \leq C(2-r)^{-2} \eta^2 \mathcal{A}_k.$$

A similar argument using  $g = [(u - l_k)_+ - R[(u - l_k)_+]]_+$  as a test function will give that

$$\|g\|_{H^s(\mathbb{R}^n)}^2 \leq C(2-r)^{-2}\eta^2 \mathcal{A}.$$

We next show a nonlinear recurrence between the  $\mathcal{A}_k$ . First, fix  $(1 - 2s)_+ < j < 1 + \gamma$ ; this is possible because if  $s \geq \frac{1}{2}$  then  $1 + \gamma > 0$ , while if  $s < \frac{1}{2}$ , we have  $1 + \gamma = \alpha + 1 - 2s > 1 - 2s$ . Then there are exponents  $t_1, t_2 \in (0, 1)$  satisfying

$$t_1 + t_2 = 1$$

such that

$$\int \{|u(x', \cdot) > l_k|\}_1^{1+\gamma} dx' \leq \left( \int \{|u(x', \cdot) > l_k|\}_1^j dx' \right)^{t_1} \|u > l_k\|_n^{t_2}.$$

This follows from applying Hölder's inequality. Beginning as usual,

$$\begin{aligned} \mathcal{A}_k &\leq C_3 C^k \left[ \int (u - l_{k-1})_+^p + \left( \int \left( \int (u - l_{k-1})_+^{2/j} dx_n \right)^j dx' \right)^{t_1} \left( \int (u - l_{k-1})_+^p \right)^{t_2} \right] \\ &\leq C_3 C^k \left[ \int v_{k-1}^p + \left( \int \left( \int v_{k-1}^{2/j} dx_n \right)^j dx' \right)^{t_1} \left( \int v_{k-1}^p \right)^{t_2} \right], \end{aligned}$$

where  $C_3 = C_2^{-\max\{p, 2t_1+pt_2\}}$ ,  $v_k = R[(w - l_k)_+] + g$ , and  $p = \frac{2n}{n-2s}$  is the fractional Sobolev embedding exponent in dimension  $n$ . The first term is clearly controlled by  $\|v_{k-1}\|_{H^s}^p$ . For the second term we apply the one dimensional Sobolev embedding and Proposition 9.2. This works because as  $j > (1 - 2s)_+$ , we have that  $\frac{2}{j} < \frac{2}{1-2s}$ , meaning that  $H^s(\mathbb{R}) \subset L^{2/j}(\mathbb{R})$ :

$$\begin{aligned} \left( \int \left( \int v_{k-1}^{2/j} dx_n \right)^j dx' \right)^{t_1} \left( \int v_{k-1}^p \right)^{t_2} &\leq C \left( \int \|v_{k-1}(x', \cdot)\|_{H^s(\mathbb{R})}^2 dx' \right)^{t_1} \|v_{k-1}\|_{H^s}^{2pt_2} \\ &\leq C \|v_{k-1}\|_{H^s(\mathbb{R}^n)}^{2t_1+pt_2}. \end{aligned}$$

The crucial observation is that as  $t_1 + t_2 = 1$ ,  $t_2 > 0$ , and  $p > 2$ , the power  $2t_1 + pt_2 > 2$ . We can now conclude that

$$\begin{aligned} \mathcal{A}_k &\leq C_3 C^k \|v_{k-1}\|_{H^s(\mathbb{R}^n)}^{p \wedge (2t_1+pt_2)} \\ &\leq C_3 C^k \left[ \|(u - l_{k-1})_+\|_{H^1(\Omega_1)} + \|g\|_{H^s(\mathbb{R}^n)} \right]^{p \wedge (2t_1+pt_2)} \\ &\leq C_3 C^k \left[ (2-r)^{-2}\eta^2 \mathcal{A}_{k-1} \right]^{\frac{p}{2} \wedge \frac{(2t_1+pt_2)}{2}}. \end{aligned}$$

Choosing  $C_2 = L(2-r)^{-1}\eta$  makes the constants universal, and so we have that for  $L$  large enough,  $\mathcal{A}_k \rightarrow 0$ , giving

$$\sup_{B_r(x_0)} u \leq l(r) + L\eta(2-r)^{-1}.$$

After optimizing in  $r$ , this implies that

$$\sup_{B_1(x_0)} u \leq C\eta^{\frac{\alpha'}{\alpha'+1}}.$$

Now applying to  $-w$  as well yields the estimate.  $\square$

## 9.4 Regularity via the Method of Campanato

In the following two sections, we present two approaches to perturbative regularity for the variable-coefficient problem. The first is a Campanato-type estimate that is well suited for situations when compatibility for the constant-coefficient approximation is not required. The argument goes as follows: in the following lemma, the approximation technique from above will be used to construct a family of functions that approximate  $u$  well at every scale and also satisfy the constant-coefficient estimates. This immediately implies some regularity of the solution near the origin, via applications of Campanato's embedding.

**Lemma 9.5.** *Let  $u$  solve (P) on  $E_2$  (with  $\Gamma = \{x_n = 0\}$ ), and assume  $a \in \mathcal{L}_2 \cap \mathcal{L}_1^*$ . Assume  $u$  is supported on  $E_3$ ,  $\|u\|_{C^{0,\alpha}(\mathbb{R}^n) \cap H^1(\Omega_1) \cap H^s(\mathbb{R}^n)} \leq 1$ , and that:*

1.  $[A]_{C^{0,\beta}(E_2)} \leq 1$
2.  $[b]_{C^{0,(\beta+1-2s)+}(E_2)} \leq 1$
3.  $\sup_{E_2} |\operatorname{div} b| \leq 1$
4.  $[f]_{C^{0,(\beta-2s)+}(\Omega_2 \cap E_2)} + [f]_{C^{0,(\beta-2s)+}(\Omega_1 \cap E_2)} + \|f\|_{L^\infty(E_2)} \leq 1$
5.  $\sup_{x,y \in E_2, z \in \mathbb{R}^n} |x-y|^{-\beta} |a^{\epsilon,i}(x, x+z) - a^{\epsilon,i}(y, y+z)| \leq 1$
6.  $\sup_{x \in E_2, z \in \mathbb{R}^n} |z|^{-\beta} |a_{a,i}(x, z)| \leq 1$

for some  $\beta < 1$ . Let  $Y_{r,x_0}$  be the following average:

$$Y_{r,x_0} = \frac{1}{r^{n-2}} \int_{B_r(x_0) \cap \Omega_1} |\nabla(u - u_{r,x_0})|^2 \\ + \frac{1}{r^{n-2s}} \int_{B_r(x_0) \times B_r(x_0)} \frac{|(u - u_{r,x_0})(x) - (u - u_{r,x_0})(y)|^2}{|x - y|^{n+2s}} dx dy$$

for every  $B_r(x_0) \subset E_1$ , where  $u_{r,x_0}$  is a solution to the following Dirichlet problem:

$$\left\{ \begin{array}{l} \forall \phi \in C_c^\infty(B_{2r}(x_0)), \quad 0 = \int_{\Omega_1} \langle A(x_0) \nabla u_{r,x_0}, \nabla \phi \rangle + \\ + \int_{\Omega_2} \int_{\Omega_1} \frac{[u_{r,x_0}(x) - u_{r,x_0}(y)] a_{s,2}(x_0, x-y) [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ + \int_{\Omega_2} \int_{\Omega_2} \frac{a_{s,1}(x_0, x-y) [u_{r,x_0}(x) - u_{r,x_0}(y)] [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ - \int u_{r,x_0} \langle b(x_0), \nabla \phi \rangle + \phi \left[ 1_{B_{2r}(x_0) \cap \Omega_1} \oint_{B_{2r}(x_0) \cap \Omega_1} f + 1_{B_{2r}(x_0) \cap \Omega_2} \oint_{B_{2r}(x_0) \cap \Omega_2} f \right] \\ \forall x \in B_{2r}^c(x_0), \quad u_{r,x_0}(x) = u(x) \end{array} \right. .$$

Then we have that

$$Y_{r,x_0} \leq C_2 r^{2\beta} \left[ \frac{1}{r^{n-2}} \|u\|_{H^1(\Omega_1 \cap B_{4r}(x_0))}^2 + \frac{1}{r^{n-2s}} \|u\|_{H^s(B_{4r}(x_0))}^2 + (\operatorname{osc}_{B_{10r}(x_0)} u)^2 + r^{2s} \right]. \quad (9.7)$$

*Proof.* Fix  $x_0$  and  $r$ , and let  $\bar{v}(x) = u(r(x - x_0) + x_0)$ . Let  $v_0$  be the solution of

$$\begin{cases} \forall \phi \in C_c^\infty(B_2(x_0)), & 0 = \int_{\Omega_1} \langle A(x_0) \nabla v_0, \nabla \phi \rangle \\ + r^{2(1-s)} \int_{\Omega_2} \int_{\Omega_1} \frac{[v_0(x) - v_0(y)] a_{s,2}^r(x_0, x-y) [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ + r^{2(1-s)} \int_{\Omega_2} \int_{\Omega_2} \frac{a_{s,1}^r(x_0, x-y) [v_0(x) - v_0(y)] [\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dy dx \\ - \int r v_0 \langle \mathbf{b}(x_0), \nabla \phi \rangle + r^2 \phi \left[ 1_{B_2(x_0) \cap \Omega_1} \int_{B_2(x_0) \cap \Omega_1} f^r + 1_{B_2(x_0) \cap \Omega_2} \int_{B_2(x_0) \cap \Omega_2} f^r \right] \\ \forall x \in B_2^c(x_0), & v_0(x) = v(x) \end{cases}$$

Set  $M = \|v\|_{H^s(B_4(x_0))} + \|v\|_{H^1(\Omega_1 \cap B_4(x_0))} + \text{osc}_{B_{10}} v + r^s$ . We claim that Lemma 9.3 applies to  $v/M, v_0/M$  with  $\eta = Cr^\beta$  (and the  $\beta$  in that lemma set to 0). Indeed, (1) – (6) scale to satisfy the hypotheses, giving that  $v - v_0$  satisfy a generalized problem. Applying Lemma 9.4 then yields

$$\|v - v_0\|_{H^s(B_1(x_0)) \cap H^1(B_1(x_0) \cap \Omega_1)} \leq MC r^\beta.$$

Then set  $u_{r,x_0}(x) = v_0(x_0 - \frac{x-x_0}{r})$  to get

$$Y_{r,x_0} \leq Cr^{2\beta} \left[ \frac{1}{r^{n-2}} \int_{B_{4r}(x_0) \cap \Omega_1} |\nabla u|^2 + \frac{1}{r^{n-2s}} \|u\|_{H^s(B_{4r}(x_0))}^2 + (\text{osc}_{B_{10r}(x_0)} u)^2 + r^{2s} \right].$$

Now it is easy to verify that  $u_{r,x_0}$  satisfies the equation promised.  $\square$

Notice that by demanding more regularity on  $f$ , the  $r^{2s}$  can be made smaller in the estimate 9.7. However, this doesn't actually improve the estimate, since the other terms on the right-hand side will be no smaller than order  $r^{2s}$ . The following is a standard lemma from calculus useful for dealing with Campanato-type arguments:

**Proposition 9.6.** *If  $\rho, \sigma$  are continuous functions  $(0, R_0) \rightarrow (0, \infty)$  satisfying  $\lim_{t \rightarrow 0^+} \sigma(t) = 0$ , and also*

$$\rho(r) \leq C \left[ \left( \frac{r}{R} \right)^\alpha + \sigma(R) \right] \rho(R)$$

*for every  $4r < R < R_0$ , then for every  $\beta < \alpha$  there is an  $R_\beta$  such that for  $r < R < R_\beta$ ,*

$$\rho(r) \leq C' \left( \frac{r}{R} \right)^\beta \rho(R).$$

*If, on the other hand,  $\rho$  is increasing and satisfies (for some  $0 < \beta < \alpha$ )*

$$\rho(r) \leq C \left[ \left( \frac{r}{R} \right)^\alpha \rho(R) + R^\beta \right]$$

*for each  $4r < R < R_0$ , then*

$$\rho(r) \leq C' \left[ \frac{\rho(R)}{R^\beta} + 1 \right] r^\beta.$$

The proof may be found in [19, Lemma 3.4]. Now we will combine the previous lemma with constant-coefficient estimates to prove bootstrap regularity,

**Theorem 9.7.** *Let  $u$  be as in Lemma 9.5 with  $\beta > 0$ . Then for every*

$$\gamma < \min\{\alpha_0(E_2), 3 - 2s, s + \beta\},$$

*we have that*

$$u \in C^\gamma(E_1 \cap \Omega_1) \cap C^\gamma(E_1 \cap \Omega_2).$$

*Also, if*

$$1 \leq \gamma < \min\{\alpha_0(E_2) \wedge 1 + (2 - 2s) \wedge \beta, s + \beta\},$$

*then*

$$u \in C^{1, \gamma-1}(E_1 \cap \Omega_1)$$

*Proof.* Assume  $\gamma \neq s, 1$ , since otherwise just prove the theorem for a slightly larger  $\gamma$ . We break the proof up into parts: first, we show  $u \in C^\gamma$  in two ways depending on the value of  $\gamma$ . Then we show the extra regularity over  $\Omega_1$ .

First, if  $\gamma < s$ , we will inductively prove that if  $u \in C^{0, k\beta/2}$  and  $\gamma' := (k/2 + 1)\beta < \min\{\alpha_0(E_2), s\}$ , then  $u \in C^{0, (k+1)\beta/2}$ . By using  $\beta$  small enough, this eventually implies  $u \in C^{0, \gamma}$ . Set

$$Z_{r,x} = \int_{B_r(x)} |u - m_{r,x}(u)|^2$$

where  $m_{r,x}(u) = \oint_{B_r(x_0)} u$ . Then we claim that if  $5r < R < 1$ ,

$$Z_{r,x} \leq C \left[ \left( \frac{r}{R} \right)^{n+2\gamma'} Z_{R,x} + R^{n+(2+k)\beta} \right].$$

Indeed, we have that (using fractional Poincaré inequality)

$$\begin{aligned} Z_{r,x} &\leq \int_{B_r(x)} |u - u_{R/5,x} - m_{r,x}(u - u_{R/5,x})|^2 + \int_{B_r(x)} |u_{R/5,x} - m_{r,x}(u_{R/5,x})|^2 \\ &\leq C \left[ \int_{B_R(x)} |u - u_{R/5,x} - m_{r,x}(u - u_{R/5,x})|^2 + r^{2\gamma'} r^n [u_{R/5,x}]_{C^{0,2\gamma'}(B_{R/5})} \right] \\ &\leq C \left[ R^n Y_{R/5,x} + r^{n+2\gamma'} [u_{R/5,x}]_{C^{0,2\gamma'}(B_{R/5})} \right]. \end{aligned}$$

The first term, using Lemma 9.5, is controlled by

$$CR^{2\beta+n} \left[ \frac{1}{R^{n-2}} \int_{\Omega_1 \cap B_{4R/5}(x)} |\nabla u|^2 + \frac{1}{R^{n-2s}} \|u\|_{H^s(B_{4R/5}(x))}^2 + R^{k\beta} + R^{2s} \right].$$

Applying the energy estimates to  $[u - m_{x,R}(u)](x_0 + \frac{4R}{5}(x - x_0))$  (with  $\phi$  a cutoff which vanishes on  $B_1(x)$  and is larger than 2 outside of  $B_2(x)$ ) and scaling back gives that the first two terms in the brackets are bounded by  $R^{-n} Z_{R,x}$ . Similarly, from the (scaled)

estimate in Theorem 8.8 and this energy argument, the other term is dominated by  $(\frac{r}{R})^{2\gamma'+n}Z_{R,x}$ . This gives

$$Z_{r,x} \leq C \left[ \left( \frac{r}{R} \right)^{n+2\gamma'} + R^{2\beta} \right] Z_{R,x} + R^{n+(2+k)\beta} \leq C \left[ \left( \frac{r}{R} \right)^{n+2\gamma'} Z_{R,x} + R^{n+(2+k)\beta} \right],$$

with the last step by the inductive assumption and the Campanato isomorphism. Applying Proposition 9.6, we obtain that

$$Z_{r,x} \leq C \left( \frac{r}{R} \right)^{(k+1)\beta+n} Z_{R,x}.$$

Applying Campanato's criterion then proves that  $[u]_{C^{0,(k+1)\beta/2}(E_1)} \leq CZ_{2,0}$ .

Now for the case  $1 > \gamma > s$ . First apply the above argument to deduce  $u \in C^{0,\gamma_1}$  for some  $\gamma_1 < s$  with  $\gamma < \gamma_1 + \beta$ . Let  $\gamma' > \gamma$  be such that a constant-coefficient estimate is available, i.e.  $\gamma' < \alpha_0(E_2) \wedge (3 - 2s)$ . Here we use the quantities

$$Z'_{r,x} = \int_{B_r(x)} |(-\Delta)^{s/2}u - m_{r,x}((-\Delta)^{s/2}u)|^2$$

which can be estimated by

$$\begin{aligned} Z'_{r,x} &\leq \int_{B_r(x)} |(-\Delta)^{s/2}(u - u_{R/5,x}) - m_{r,x}((-\Delta)^{s/2}(u - u_{R/5,x}))|^2 + \\ &\quad + \int_{B_r(x)} |(-\Delta)^{s/2}u_{R/5,x} - m_{r,x}((-\Delta)^{s/2}u_{R/5,x})|^2 \\ &\leq CR^{n-2s}Y_{R/5,x} + C \left( \frac{r}{R} \right)^{n+2(\gamma'-s)} \int_{B_{R/5}(x)} |(-\Delta)^{s/2}u_{R/5,x} - m_{R/5,x}((-\Delta)^{s/2}u_{R/5,x})|^2 \\ &\leq CR^{n-2s+2\beta} \left[ \frac{1}{R^{n-2}} \int_{\Omega_1 \cap B_{4R/5}(x)} |\nabla u|^2 + \frac{1}{R^{n-2s}} \|u\|_{\dot{H}^s(B_{4R/5}(x))}^2 + R^{2\gamma_1} \right] + \\ &\quad + C \left( \frac{r}{R} \right)^{n+2(\gamma'-s)} \left[ Z'_{R,x} + \int_{B_{R/5}(x)} |(-\Delta)^{s/2}(u - u_{R/5,x})|^2 \right] \\ &\leq CR^{n+2(\gamma_1-s)+2\beta} + C \left( \frac{r}{R} \right)^{n+2(\gamma'-s)} [Z'_{R,x} + R^{n+2(\gamma_1-s)+2\beta}] \\ &\leq CR^{n-2s+2(\gamma_1+\beta)} + C \left( \frac{r}{R} \right)^{n+2(\gamma'-s)} Z'_{R,x}, \end{aligned}$$

where we used the  $C^{\gamma'}$  constant coefficient estimate, the fact  $u \in C^{0,\gamma_1}$ , the energy estimate as above, and Campanato's criterion. Applying Proposition 9.6 gives

$$Z'_{r,x} \leq C \left( \frac{r}{R} \right)^{2(\gamma-s)+n} Z_{R,x},$$

and so the conclusion follows from another application of Campanato's criterion, which shows that  $(-\Delta)^{s/2}u \in C^{\gamma-s}(E_1)$ , and then regularity for the fractional Laplace equation. If  $\gamma > 1$ , the same works with  $m_{r,x}$  redefined to be

$$1_{\Omega_1} \int_{\Omega_1 \cap B_r(x)} \cdot + 1_{\Omega_2} \int_{\Omega_2 \cap B_r(x)} \cdot.$$



To prove that  $u \in C^{1,\gamma-1}(E_1 \cap \Omega_1)$  we proceed as follows. Let  $\gamma_1 < \alpha_0(E_2)$  be such that  $\gamma_1 + \beta > \gamma$ , and  $\gamma < \gamma' < \alpha_0(E_2) \wedge 1 + 2 - 2s$ . Take the modified quantities

$$Z_{r,x}^1 = \int_{B_r(x) \cap \Omega_1} |\nabla u - m_{r,x}(\nabla u)|^2,$$

Proceed by estimating

$$\begin{aligned} Z_{r,x}^1 &\leq \int_{B_r(x) \cap \Omega_1} |\nabla u - \nabla u_{R/5,x} - m_{r,x}(\nabla u - \nabla u_{R/5,x})|^2 \\ &\quad + \int_{B_r(x) \cap \Omega_1} |\nabla u_{R/5,x} - m_{r,x}(\nabla u_{R/5,x})|^2 \\ &\leq C \left[ \int_{B_R(x) \cap \Omega_1} |\nabla u - \nabla u_{R/5,x} - m_{r,x}(\nabla u - \nabla u_{R/5,x})|^2 \right. \\ &\quad \left. + \left(\frac{r}{R}\right)^{n+2\gamma'-2} \int_{B_{R/5}(x) \cap \Omega_1} |\nabla u_{R/5,x} - m_{R/5,x}(\nabla u_{R/5,x})|^2 \right] \\ &\leq C \left[ R^{n-2} Y_{R/5,x} + \left(\frac{r}{R}\right)^{n+2\gamma'-2} \left( Z_{R,x}^1 + \int_{B_{R/5}(x) \cap \Omega_1} |\nabla(u - u_{R/5,x})|^2 \right) \right] \\ &\leq C \left[ R^{n-2+2\beta+2\gamma_1} + \left(\frac{r}{R}\right)^{n+2\gamma'-2} Z_{R,x}^1 \right], \end{aligned}$$

using the energy estimate for  $u$  in the last step. Proceeding as before, we deduce that  $u \in C^{1,\gamma-1}(E_1 \cap \Omega_1)$ .  $\square$

## 9.5 Regularity via Improvement of Flatness

In this section we give a more localized perturbative framework which will give stronger conclusions than the previous method. The most obvious advantage to this approach is the near-optimal regularity on  $\Omega_2$  in the vicinity of each point of the interface, including the case of compatible coefficients. The majority of the work is condensed into the following lemma.

**Lemma 9.8.** *Let  $u$  solve (P) on  $E_2$  with  $a \in \mathcal{L}_2 \cap \mathcal{L}_1^*$ ,  $\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq 1$ . Select a  $\gamma$  with  $(2s-1)_+ < \gamma \leq 1$  and  $\gamma < \alpha_0(0)$ . Also assume the following:*

1.  $[A]_{C^{0,\beta}(E_2)} \leq 1$
2.  $[b]_{C^{0,(\beta+1-2s)_+}(E_2)} \leq 1$
3.  $\sup_{E_2} |\operatorname{div} b| \leq 1$
4.  $[f]_{C^{0,(\beta+\gamma-2s)_+}(\Omega_2 \cap E_2)} + [f]_{C^{0,(\beta+\gamma-2s)_+}(\Omega_1 \cap E_2)} + \|f\|_{L^\infty(E_2)} \leq 1$
5.  $\sup_{x,y \in E_2, z \in \mathbb{R}^n} |x-y|^{-\beta} |a^{\epsilon,i}(x, x+z) - a^{\epsilon,i}(y, y+z)| \leq 1$
6.  $\sup_{x \in E_2, z \in \mathbb{R}^n} |z|^{-\beta} |a_{a,i}(x, z)| \leq 1$ .

Then for every  $\beta' < \beta$  there exists an  $r > 0$  such that

$$\sup_{B_{r^k}} |u - v_k| \leq C_0 r^{(\gamma+\beta')k},$$

where  $v_k$  is a sum of scaled solutions to constant-coefficient equations

$$v_k(x) = \sum_{j=1}^k r^{(\gamma+\beta')(j-1)} w_j(r^{-j}x),$$

each of which satisfy the estimates

$$\|w_j\|_{C^\gamma(B_1 \cap \Omega_2) \cap C^{1,\gamma+1-2s}(B_1 \cap \Omega_1)} + \sum_{i,l \neq n} \|\partial_{e_i e_l} w_j\|_{C^{0,\alpha}(B_1)} + \| |x_n|^{\gamma-1} \nabla w_j \|_{L^\infty} \leq C_0.$$

A couple of remarks are in order. In this lemma,  $\gamma$  should be thought of as being close to the minimum of  $\alpha_0(0)$  and 1, and represents the regularity gained “for free”, regardless of the quantitative assumptions on the coefficients. In classical variational Schauder theory, if the coefficients are in any Hölder space, regardless of the exponent, the solution will be  $C^1$ , and our  $\gamma$  plays a role analogous to this 1. Perhaps surprisingly,  $\gamma$  may exceed  $s$  in some situations. Notice also that all of the approximating functions solve a constant-coefficient equation *frozen at the same point*; this is a key improvement over the previous section.

*Proof.* Before commencing, observe that we may assume (by making  $r$  small and dilating) that the conditions (1) – (6) are satisfied with  $\eta$  rather than 1, for some small  $\eta$  to be chosen below.

The proof is by induction on  $k$ , with the extra hypotheses that  $w_k$  solves the Dirichlet problem

$$\begin{cases} B_L^{(0)}[w_k, \phi] + r^{2k(1-s)} B_N^{r^k, (0)}[w_k, \phi] \\ \quad = r^k \int w_k \langle \mathbf{b}(0), \nabla \phi \rangle \\ \quad \quad + r^{2k} \int \phi \left[ 1_{\Omega_1} \int_{B_3 \cap \Omega_1} f + 1_{\Omega_2} \int_{B_3 \cap \Omega_2} f \right] \\ w_k = r^{-(k-1)(\gamma+\beta')} (u - v_{k-1})(r^k x) \end{cases} \quad \begin{array}{l} \phi \in C_c^\infty(B_3) \\ x \notin B_3 \end{array},$$

where  $A^{(0)} = A(0)$ ,  $a_{s,i}^{r^k, 0}(x, z) = a_{s,i}(0, r^k z)$ , and  $B_L^{(0)}, B_N^{r^k, (0)}$  are the corresponding forms, while  $u - v_k$  solves a generalized problem

$$\begin{cases} B_L[u - v_k, \phi] + B_N[u - v_k, \phi] \\ \quad = \int \left( f - \left[ 1_{\Omega_1} \int_{B_3 \cap \Omega_1} f + 1_{\Omega_2} \int_{B_3 \cap \Omega_2} f \right] \right) \phi \\ \quad \quad + (u - v_k) \langle \mathbf{b}, \nabla \phi \rangle + \int_{\Omega_1} \langle f_1, \nabla \phi \rangle \\ \quad \quad + \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{h(x, y)[\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dx dy \\ u - v_k = 0 \end{cases} \quad \begin{array}{l} \phi \in C_c^\infty(B_{3r^k}) \\ x \notin B_{3r^k} \end{array}$$

where  $r^{k(1-\gamma)}|f_1| \leq \eta(1-t^k)|x|^{\beta'}$ ,  $|h(x, y)| \leq \eta|x-y|^\gamma(1-t^k)(|x|^{\beta'} + |y|^{\beta'})$ , and  $|h(x, y)| \leq \eta|x-y|(1-t^k)(|x|^{\beta'} + |y|^{\beta'})|x_n|^{\gamma-1}$  for  $|x-y| < \frac{1}{2}|x_n|$ , where  $t > 0$  is some small constant to be determined below and  $x, y \in B_{3r^{k+1}}$ .

Assume this holds for the first  $k-1$  iterations; we show it holds for the  $k$ th. Write  $\bar{w}_j(x) = r^{-(\gamma+\beta')(k-1)}w_j(r^k x)$ , and similarly for  $\bar{v}$  and  $\bar{u}$ . The idea is that  $\bar{u} - \bar{v}_{k-1}$  solves a generalized problem in the sense of Lemma 9.3, while  $\bar{w}_k$  is the corresponding constant-coefficient solution.

Indeed, from scaling we have that

$$\begin{cases} B_L^k[\bar{u} - \bar{v}_{k-1}, \phi] + r^{2k(1-s)}B_N^k[\bar{u} - \bar{v}_{k-1}, \phi] = \int r^{2k-(\gamma+\beta')(k-1)}f^{r^k}\phi \\ \quad + r^k(\bar{u} - \bar{v}_{k-1})\langle \mathbf{b}^{r^k}, \nabla \phi \rangle + \int_{\Omega_1} r^{k-(\gamma+\beta')(k-1)}\langle f_1^{r^k}, \nabla \phi \rangle & \phi \in C_c^\infty(B_{3/r}) \\ \quad + r^{2k(1-s)-(\gamma+\beta')(k-1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{h^{r^k}(x, y)[\phi(x) - \phi(y)]}{|x-y|^{n+2s}} dx dy \\ u - v_{k-1} = 0 & x \notin B_{3/r} \end{cases}.$$

We check that all the hypotheses of Lemma 9.3 are met on  $B_3$ . The ones on the coefficients,  $f$ , and  $\mathbf{b}$  are immediate (using here the regularity assumption on  $f$  and the fact that  $\gamma < 2s$ ). We also have that for  $f_1$ ,

$$r^{k-(\gamma+\beta')(k-1)}|f_1^{r^k}| \leq \eta r^{(\gamma-1)k} r^{k-(\gamma+\beta')(k-1)+k\beta'}|x|^{\beta'} \leq \eta r^{\gamma+\beta'}|x|^{\beta'},$$

while for  $h$ ,

$$r^{-(\gamma+\beta')(k-1)}|h^{r^k}(x, y)| \leq r^{-(\gamma+\beta')(k-1)}\eta r^\gamma|x-y|^\gamma r^{\beta'}(|x|^{\beta'} + |y|^{\beta'}) \leq \eta r^{\gamma+\beta'}|x-y|^\gamma(|x|^{\beta'} + |y|^{\beta'}),$$

and similarly for the other condition. By choosing  $\eta$  small enough, we can then arrange that  $w_k$  satisfies

$$\sup_{B_1} |w_k - (\bar{u} - \bar{v}_{k-1})| \leq r^{\gamma+\beta'},$$

and hence

$$\sup_{B_{r^k}} |v_k - u| \leq C_0 r^{(\gamma+\beta')k}.$$

The interior estimates on  $w_k$  are a consequence of the previous section. We must check that  $u - v_k$  solves the corresponding generalized problem. This comes from applying Lemma 9.3 choosing  $r$  small, and scaling back (using  $f_1'$  for the new  $f_1$  in the equation for  $u - v_k$  and  $\tilde{f}_1$  for the change in generalized right-hand side in the scaled problem):

$$\begin{aligned} |f_1'(x)| &\leq |f_1| + |r^{(\gamma+\beta')(k-1)}\tilde{f}_1(r^{-k}x)| \\ &\leq \eta(1-t^{k-1})|x|^{\beta'} + \eta C r^{k(\gamma+\beta'-1)}|r^{-k}x|^{\beta} \\ &\leq \eta(1-t^{k-1})|x|^{\beta'} + \eta C r^{(\beta-\beta')}|x|^{\beta'} \\ &\leq \eta|x|^{\beta'}(1-t^k) \end{aligned}$$

provided  $r$  is small enough and  $|x| \leq 3r^{k+1}$ , while

$$\begin{aligned} |h'(x, y)| &= |h(x, y)| + r^{(\gamma+\beta')(k-1)} |\tilde{h}'(r^{-k}x, r^{-k}y)| \\ &\leq \eta(1 - t^{k-1})|x - y|^\gamma(|x|^{\beta'} + |y|^{\beta'}) + \eta r^{k\beta'} |x - y|^\gamma(|r^{-k}x|^\beta + |r^{-k}y|^\beta) \\ &\leq \eta(1 - t^k)|x - y|^\gamma(|x|^{\beta'} + |y|^{\beta'}). \end{aligned}$$

The other estimate works the same. and this completes the argument.  $\square$

**Corollary 9.9.** *Let  $u$  be an admissible solution to (P) on  $E_2$  with  $a \in \mathcal{L}_2 \cap \mathcal{L}_1^* \cap \mathcal{A}_1 \cap C_t^{0,1}$ ,  $A \in C^{0,1}$ ,  $\mathbf{b} \in C^{0,1}$ , and  $\Gamma \in C^{1,1}$ . Then for each  $\gamma < \alpha_0(E_2) \wedge (3 - 2s)$ ,  $u \in C^\gamma(\Omega_1 \cap E_1) \cap C^\gamma(\Omega_2 \cap E_1)$ . Also, for each  $(2s - 1)_+ < \gamma < \alpha_0 \wedge 1$ ,  $u \in C^{1, \gamma+(1-2s)-}(\Omega_1 \cap E_1)$ , and  $\partial_e u \in C^{0, \gamma}(E_1)$  for every  $e \perp e_n$ . If  $a_{s,i}^{(0)}(0, \cdot), A(0)$  are compatible and  $\alpha_0(0) > 1$ , there are  $v_1, v_2$  such that for each  $\delta$ ,*

$$u(x) = \begin{cases} u(0) + \langle x, v_1 \rangle + O(|x|^{\alpha_0(0)+2s-2-\delta}) & x \in \Omega_1 \\ u(0) + \langle x, v_2 \rangle + O(|x|^{\alpha_0(0)-\delta}) & x \in \Omega_2 \end{cases}.$$

Finally,  $u$  is the unique admissible solution to (P) with this boundary data.

*Proof.* (sketch) Apply the boundary straightening described in 9.1 to  $u$  (centered on 0), and then apply the lemma above. Set  $\beta = 1$  and note that

$$\sup_{B_{r^k}} |v_k(x) - v_k(0)| \leq Cr^{k(\gamma \wedge 1)},$$

$$\sup_{B_{r^k} \cap \Omega_2} |v_k(x) - v_k(0) - \langle \nabla^+ v_k(0), x \rangle| \leq Cr^{k\gamma}$$

where the second holds for  $\gamma > 1$  and  $\nabla^+$  means computed from  $\Omega_2$ . Also

$$\sup_{B_{r^k} \cap \Omega_1} |v_k(x) - v_k(0) - \langle \nabla^- v_k(0), x \rangle| \leq Cr^{k(\gamma+2-2s)},$$

which implies the first and last conclusions. For the tangential regularity, use that

$$\sup_{B_{r^k}} |v_k(x', x_n) - v_k(0, x_n) - \langle \nabla_{x'} v_k(0, x_n), x' \rangle| \leq Cr^2.$$

To pass from point estimates at 0 to Hölder estimates in neighborhoods, flatten around each  $x \in \Gamma \cap E_1$  and proceed the same way, finally combining with interior estimates. To prove uniqueness, proceed as in 8.9, using that under the assumptions of the corollary, interior estimates place  $u \in W^{1,p}$  for some  $p$ .  $\square$

The corollary above is only a sample of what can be shown with this method, and can clearly be generalized substantially. Indeed, the lemma in this subsection asserts that to order  $\beta + \alpha_0 \wedge 1$ , a solution to the variable-coefficient problem coincides with a superposition of solutions to the constant-coefficient problem frozen at the point in question. Except perhaps in the event that  $2 - 2s$  is very large (but see the remark below), this fully resolves the regularity issue for variable-coefficient problems.

## 9.6 A Remark On Higher Regularity on $\Omega_1$

In the arguments above, there was a limit to how much regularity could be proved in the normal direction: it was capped by  $\beta + \alpha_0 \wedge 1$ . On  $\Omega_2$ , this is never an issue, as  $\alpha_0$  is already the maximum expected regularity for the equation, and is less than 2. On  $\Omega_1$ , however, this can fall short of the expected estimate  $\alpha_0 + 2 - 2s$  (basically when  $\alpha_0$  is small and  $s < \frac{1}{2}$ ). This can be circumvented easily as follows: the estimates above do show (for  $\beta$  large enough) that  $A\nabla^-u(x', 0) = 0$ . But now purely local Schauder theory can be applied, treating this as a second-order linear divergence-form Neumann problem. For instance, the argument given in Sections 7 and 8 goes through using properties of the fundamental solution of  $\operatorname{div} A \nabla \cdot = 0$ . Note that as soon as  $u \in C^{0,\alpha}(E_1)$  for some  $\alpha > 2s - 1$ , the energy argument for the distributional vanishing of the conormal can be applied instead (i.e. the proof of Lemma 7.6 goes through).

## 9.7 Applications to the Case of Nonlinear Drift

The above theory already covers the case of critical (SQG-type) drift; we briefly explain how it can be applied to this context.

**Corollary 9.10.** *There exists a unique admissible solution  $u$  of the following, for  $\Gamma \in C^{1,1}$  globally (uniformly) and satisfying Condition 5.1:*

$$\begin{cases} \forall \phi \in C_c^\infty(\mathbb{R}^n), & \int_{\Omega_1} \langle \nabla u, \nabla \phi \rangle = \int u \langle \mathbf{b}, \nabla \phi \rangle + f \phi \\ & - \int \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{[\nu 1_{\Omega_1 \times \Omega_2} + \nu 1_{\Omega_2 \times \Omega_1} + 1_{\Omega_2 \times \Omega_2}][u(x) - u(y)][\phi(x) - \phi(y)]}{|x - y|^{n+2s}} dx dy \\ \mathbf{b} = T u \end{cases}$$

where  $T = G\vec{R}u$  for  $G$  a fixed skew-symmetric matrix and  $\vec{R}$  the vector of Riesz transforms. Assume  $f \in C_c^\infty$  and  $\nu > 0$ . Then  $u$  is in  $C^{0,\gamma}(\mathbb{R}^n)$  for every  $\gamma < \alpha_0$ , where  $\alpha_0$  depends only on  $\nu$  and  $\|f\|_{L^\infty \cap L^1}$ . Moreover,  $u \in C^{1,\gamma}(\Omega_1)$ .

*Proof.* (sketch) First apply Lemma 5.4 to obtain a bounded admissible solution tot his problem, and then Theorem 5.5 see that  $u \in C^{0,\alpha}(\mathbb{R}^n)$  for some  $\alpha > 0$ . From the basic theory of singular integrals, it follows that  $\mathbf{b} \in C^{0,\alpha}$  as well. Now apply Theorem 9.7 to  $u$  after flattening the boundary (locally at each point  $x \in \Gamma$ ). Then the regularity of  $u$  increases to any Hölder space with exponent less than  $(\alpha + \frac{1}{2}) \wedge \alpha_0$  (where  $\alpha_0 \in (0, 1)$  is determined as in Theorem 8.8). Re-estimate the Hölder regularity of the drift, and reapply the Theorem to give  $u \in C^{0,\gamma}(E_1)$ . Finally apply Section 9.6 to conclude that  $u \in C^{1,\gamma}(\Omega_1)$ .

This leaves only the uniqueness assertion. From interior regularity, we can obtain the following weighted estimate on  $u$ :

$$\sup_{\Omega_2} |x_n|^{\alpha-1} |\nabla u| \leq C.$$

This implies that  $u \in W_{\text{loc}}^{1,p}$  for some  $p > 1$ ; see Proposition 8.9 for details. The same holds for any other admissible solution  $v$ . Writing the weak-form equation for  $w = u - v$ ,

$$B_L[w, \phi] + B_N[w, \phi] = \int \phi \langle Tu, \nabla u \rangle - v \langle Tv, \nabla \phi \rangle.$$

Letting  $\phi_k \rightarrow w$  in  $W_{\text{loc}}^{1,p} \cap H^s \cap H^1(\Omega_1)$ , observe that the right-hand side tends to 0, giving that  $w$  is identically 0.  $\square$

## Appendix

Here we gather several results used above; the proofs are mostly elementary and are included for completeness.

**Lemma 2.1.** *There is a bounded linear operator  $T : V \rightarrow H_0^1(E_1)$ , where  $V$  is the closure of  $\{u \in C^\infty(\Omega_1 \cap E_1) : u|_{\partial E_1 \cap \Omega_1} = 0\}$  in  $H^1(\Omega_1 \cap E_1)$ , with  $\|T\|$  depending only on  $L$ , satisfying the following properties:*

1.  $Tv|_{\Omega_1 \cap E_1} = v$  a.e.
2. If  $v \geq 0$ ,  $Tv \geq 0$ .
3. For every  $v \in V$  and  $l > 0$ ,

$$|\{Tv > l\}| \leq \frac{4}{3} |\{v > l\} \cap \Omega_1 \cap E_1|.$$

*Proof.* Denote by  $E$  the infinite cylinder  $\{|x'| < 1\}$ ; for convenience assume  $\Gamma \cap E$  is contained in the graph of  $g$ . Extend  $v$  by 0 to  $E \cap \Omega_1$ , and define  $Tv$  as follows:

$$Tv(x', x_n) = \begin{cases} v(x', x_n) & x_n \geq g(x') \\ v(x', 4g(x') - 3x_n) & x_n < g(x') \end{cases}.$$

Clearly  $Tv$  is linear and preserves positivity. A standard computation reveals that

$$\nabla Tv(x', x_n) = \begin{cases} \nabla v(x', x_n) & x_n \geq g(x') \\ \begin{pmatrix} 1 & 0 & \cdots & 4\partial_1 g(x') \\ 0 & 1 & \cdots & 4\partial_2 g(x') \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -3 \end{pmatrix} \nabla v(x', 4g(x') - 3x_n) & x_n < g(x') \end{cases}$$

is in fact a weak derivative for  $Tv$ , so from change of variables formula we easily deduce

$$\|Tv\|_{H^1(E)} \leq C(n, L) \|v\|_{H^1(E \cap \Omega_1)}.$$

From the definition of  $T$ , it is simple to check that  $Tv(x', x_n) = 0$  for almost every  $x_n \leq -2L$ , meaning  $Tv \in H_0^1(E_1)$ . Finally, we compute the measure of level sets:

$$\begin{aligned}
| \{Tv > l\} | &\leq | \{v > l\} \cap \Omega_1 | + | \{Tv > l\} \cap \Omega_2 | \\
&= \int_{\{v > l\} \cap \Omega_1} 1 + \left| \det \begin{pmatrix} 1 & 0 & \cdots & 4\partial_1 g(x') \\ 0 & 1 & \cdots & 4\partial_2 g(x') \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -3 \end{pmatrix} \right|^{-1} \\
&= \frac{4}{3} | \{v > l\} \cap \Omega_1 |,
\end{aligned}$$

where we've used the fact that  $g$  is Lipschitz and the area formula.  $\square$

**Lemma 2.3.** *Assume  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $|u| \leq 1 + |x|^\alpha$  for some  $\alpha < \gamma$ . Then there are constants  $\{c_B\}$  such that*

$$\sup_{B \subset B_1} \frac{1}{|B|} \int_B |Tu - c_B| \leq C,$$

where  $C$  depends only on  $T$  and  $\gamma - \alpha$ .

*Proof.* Fix  $B$  and let  $B^*$  be its double. Write  $u = u_1 + u_2 = 1_{B^*}u + 1_{B^{*c}}u$ . Then

$$\frac{1}{|B|} \int_B |Tu_1| \leq \left( \frac{1}{|B|} \int_B |Tu_1|^2 \right)^{1/2} \leq C \left( \frac{1}{|B|} \int_B u_1^2 \right)^{1/2} = C \left( \frac{1}{|B^*|} \int_{B^*} u^2 \right)^{1/2} \leq C,$$

where the second inequality used the boundedness of  $T$  on  $L^2$ , while the last that  $B^* \subset B_2$  and the bound on  $u$ . On the other hand, if  $B = B_r(x_0)$ , set

$$c_B = \int_{\mathbb{R}^n \setminus B^*} K(x_0 - y)u(y)dy.$$

Then we can estimate

$$\begin{aligned}
\int_B |Tu_2(x) - c_B|dx &\leq \int_B \int_{\mathbb{R}^n \setminus B^*} |K(x - y) - K(x_0 - y)| |u(y)| dy dx \\
&\leq C \int_B \int_{\mathbb{R}^n \setminus B^*} \frac{|x - x_0|^\gamma}{|y - x_0|^{n+\gamma}} |u(y)| dy dx \\
&\leq C r^{n+\gamma} \int_{\mathbb{R}^n \setminus B^*} \frac{1 + |y - x_0|^\alpha}{|y - x_0|^{n+\gamma}} dy \\
&\leq C r^{n+\gamma} (1 + r^{-\gamma}) \leq C|B|,
\end{aligned}$$

where the second line uses (2.3).  $\square$

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